Discretization w.r.t. structural properties

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Outline

Reminder on **spatial discretization**

Notion on **port-based modelling**

1D case discretization of port Hamiltonian system

Finite elements method: **Whitney elements**

Ojectives

Spacial discretization

Time-invariant continuous problem in D: given μ, find b, h such that

$$(1)\ \mathrm{rot}\, h = 0, [n \times h] = 0, \quad (2)\ b = \mu h, \quad (3)\ \mathrm{div}\, b = 0\,, [n \cdot b] = 0$$

Equivalent formulation

decompose D and consider

$\Phi^I = \{\varphi = 0 \text{ on } S_0^h \text{ and } \varphi = I \text{ on } S_1^h\}$

notice $\varphi \in \Phi^I \Rightarrow (1)$ with $h = \mathrm{grad}\,\varphi$

if furthermore

$\int_D \mu\, \mathrm{grad}\varphi \cdot \mathrm{grad}\varphi' = 0\,, \forall \varphi' \in \Phi^0$

then $b = \mu\, \mathrm{grad}\varphi = \mu h$ satisfies (3)

continuous formulation

Ritz-Galerkin method

finite number of elements $\lambda^i \in \Phi$ called **trial functions**

idea: reach the field of least energy $\sum_{\in \mathcal{I}} \varphi_i \lambda^i \in \Phi^I$

define $\Phi_m^I = \mathrm{span}\, \{\lambda^i\} \bigcap \Phi^I$

and find $\varphi_m \in \Phi_m^I$ such that

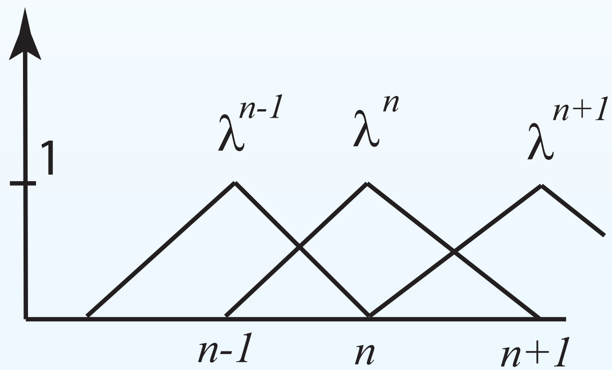
$\int_D \mu\, \mathrm{grad}\varphi_m \cdot \mathrm{grad}\varphi' = 0\,, \forall \varphi' \in \Phi_m^0$

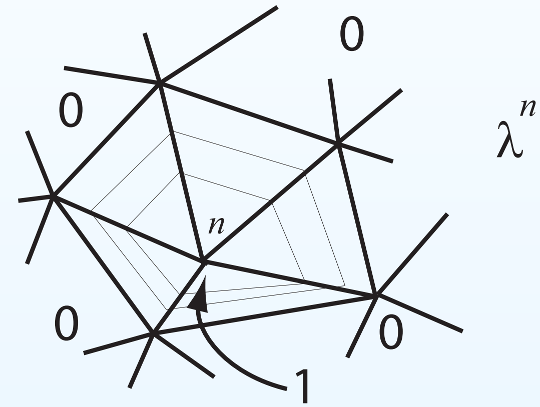
discrete formulation

variational formulation: find $\varphi_m \in \Phi_m^I$ s.t. $F(\varphi_m) \leq F(\psi)\,, \forall \psi \in \Phi_m^I$, with $F(\varphi, \psi) = \int_D \mu \nabla\varphi \cdot \nabla\psi$

Finite elements

- **paving** spatial domain D: tetrahedra, triangles, ...
- **nodal functions**: λ^i defined on the meshing (define a partition if unity $\sum_{n\in\mathcal{N}} \lambda^n = 1$)





Consider now the set $\Phi_m = \{\varphi = \sum_{n\in\mathcal{N}} \varphi_n \, \lambda^n\}$

Computation:

setting $M_{n,m} = \int_D \mu \, \mathrm{grad}\lambda^n \cdot \mathrm{grad}\lambda^m$ it follows

$$\int_D \mu \, \nabla\varphi_m \cdot \nabla\varphi'_m = 0 \Leftrightarrow \langle M\varphi_m, \varphi'_m\rangle = 0 \quad \forall \varphi'_m \in \Phi^0_m$$

reduce to a standard **linear** system

Port-based approach

- based on the **energy** and **port conjugate variables**
- leads to modelling systems according to **effort-flux** variables
- provides **geometric** characterization of **properties** (*e.g.* energy-conservation, symmetries)

Dynamics of Maxwell's equation $Z \subset \mathbb{R}^3$

energy function $H = \int_Z \mathcal{H}(d, b)$
energy variables: d magnetic induction, b electric induction
co-energy variables: $e = \delta_d H$ electric field, $h = \delta_b H$ magnetic field

Ampère's thm: $\dot{d} = \mathrm{d}h$ Faraday's law: $\dot{b} = -\mathrm{d}e$

$$\begin{pmatrix} -\dot{d} \\ -\dot{b} \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{d} \\ \mathrm{d} & 0 \end{pmatrix} \begin{pmatrix} \delta_d H \\ \delta_b H \end{pmatrix}, \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} \delta_d H_{|\partial Z} \\ \delta_b H_{|\partial Z} \end{pmatrix}$$

Stokes-Dirac structure

in particular, **intrinsically** $\dot{H} = \int_{\partial Z} h \wedge e$

Mixed finite elements

idea: take **two** basis of nodal functions, one basis to approximate **fluxes** and one basis to approximate **efforts**

Example of the transmission line $Z = [0, L] \subset \mathbb{R}$

$$\dot{q}(t,z) = -\partial_z I(t,z)\,, \qquad \dot{\phi}(t,z) = -\partial_z V(t,z)$$

equivalently $\begin{pmatrix} -\dot{q} \\ -\dot{\phi} \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{d} \\ \mathrm{d} & 0 \end{pmatrix} \begin{pmatrix} \delta_q H \\ \delta_\phi H \end{pmatrix}$

discretization:
let $0 \leq a < b \leq L$ and denote $Z_{ab} = [a, b]$

approximation of flux: $f(t,z) = f_{ab}(t) w_{ab}(z)$

approximation of effort: $e(t,z) = e_a(t) w_a(z) + e_b(t) w_b(z)$

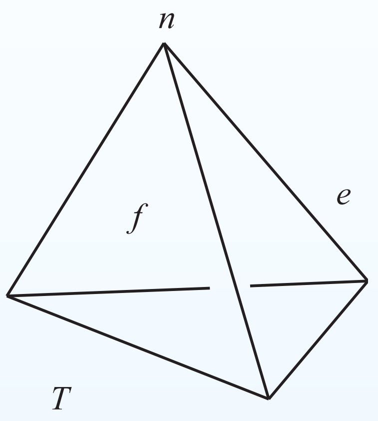
conjugate variables (f_{ab}, e_{ab}) derived from power balance satisfy

$$(f_{ab}, e_{ab}) \in \mathcal{D}_{ab}$$

discrete formulation satisfies the energy balance

Whitney elements

Specific nodal functions on tetrahedra



satisfying

- w_n takes value 1 at node n (0 elsewhere)
- the circulation of w_e along edge e is 1
- the flux of w_f across face f is 1
- the integral of w_T over a tetrahedra T is 1

ad hoc properties for discretization (approximation, topology, ...)

efficiency for computation

Objectives

Combine **port-based modelling** and **mixed Whitney elements** for the 3D discretization and simulation of Maxwell's equation

- definition of **port variables** on T
- define the **interconnection** between elements prolonging properties to the discrete formulation
- simplifications (why not?)
- implementation
- higher order splines: second degree polynomials $\lambda^n \lambda^m$

- **modularity**: reproductible method
- **boundaries**: port variables suitable to represent interaction with environment
- **multidomain**: interconnection structure