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Modern Physics: THE SYMMETRY PRINCIPLE

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Symmetries	$\Leftrightarrow$	**conserved quantities**
Fields	$\Leftrightarrow$	**momentum $\approx$ derivatives**
Wave equation	$\Leftrightarrow$	**relativity**
Charge conservation	$\Leftrightarrow$	**gauge principle**
Symmetry breaking	$\Leftrightarrow$	**buckling**

The Lagrangian is independent of time:

$$0 =_{\text{chain}} \frac{d}{dt}\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}}\dot{\mathbf{q}} - \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\ddot{\mathbf{q}} =_{\text{E.L.}} \frac{d}{dt}\left(\dot{\mathbf{q}}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \mathcal{L}\right) = \frac{d}{dt}\mathcal{H}$$

conservation of energy

where $\mathcal{H}$ = Hamiltonian, and Euler-Lagrange equations:

$$0 = \frac{\partial \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}})}{\partial \mathbf{q}} - \frac{d}{dt}\frac{\partial \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}}$$

If the Lagrangian is not a function of position $\mathbf{q}$:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{q}}} = 0 \quad \Leftrightarrow \quad \frac{\partial\mathcal{L}}{\partial\dot{\mathbf{q}}} = \text{constant} = \mathbf{p}$$

conservation of momentum

TWO-BODY (angular momentum, $\mathbf{q}_i, \mathbf{p}_i \in \mathbb{R}^3$)

$$\mathcal{L} = \frac{m_1 \dot{\mathbf{q}}_1^2}{2} + \frac{m_2 \dot{\mathbf{q}}_2^2}{2} - V(|\mathbf{q}_1 - \mathbf{q}_2|)$$

translation of the whole system (position)

$$0 = \sum_i \frac{\partial \mathcal{L}}{\partial \mathbf{q}_i} = \frac{d}{dt} \sum_i \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}_i} = \frac{d}{dt}(\mathbf{p}_1 + \mathbf{p}_2)$$

rotation of the whole system (position and velocity)

$$0 = \sum_i \frac{\partial \mathcal{L}}{\partial \mathbf{q}_i} \cdot (\mathbf{q}_i \times \mathbf{n}) + \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}_i} \cdot (\dot{\mathbf{q}}_i \times \mathbf{n}) = \mathbf{n} \cdot \frac{d}{dt} \left(\mathbf{q} \times \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}_i} \right)$$

Digression

The Poisson bracket $\{\cdot,\cdot\}$:

$$\{f,g\} = \frac{\partial f}{\partial \mathbf{p}}\frac{\partial g}{\partial \mathbf{q}} - \frac{\partial g}{\partial \mathbf{p}}\frac{\partial f}{\partial \mathbf{q}}$$

(such that: $\{p_i, q_j\} = \delta_{ij}$ and $\dot{g} = \{H, g\}$) led Dirac (1925) to a simple definition of Quantum Mechanics:

$\{f,g\} \to [F,G] = FG - GF$ that satisfy $[P_i, Q_j] = i\hbar\delta_{ij}$

every operator $F(P,Q), G(P,Q)$, and $P(Q)$ or $Q(P)$.

FIELDS: *infinite number of degrees of freedom*, $\mathbf{q}$

$$\mathbf{q}_i \rightarrow \phi(\mathbf{q},t) \qquad \mathbf{p}_i \rightarrow \pi(\mathbf{q},t) = \frac{\delta\mathcal{L}(\phi(\mathbf{q}),\dot{\phi}(\mathbf{q}))}{\delta\dot{\phi}(\mathbf{q})}$$

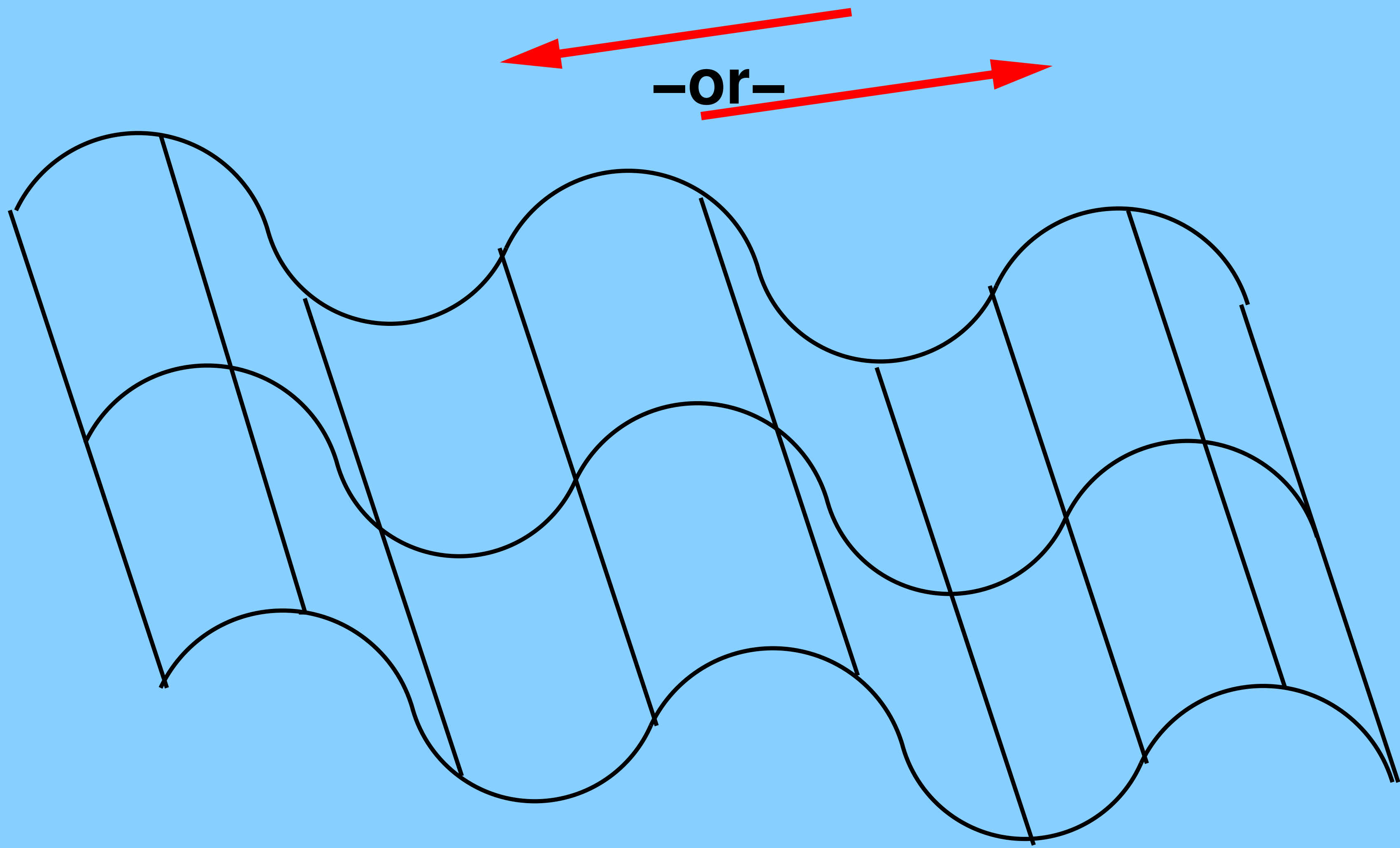
momentum also as translational invariance:

$$\phi(\mathbf{q}+\mathbf{a}) = \phi(\mathbf{q}) + \mathbf{a}\frac{\partial\phi(\mathbf{q})}{\partial\mathbf{q}} + \cdots$$

second type of "momentum density" with $\partial_\mu P^\mu = 0$:

$$\mathbf{P}(\mathbf{q}) = -\pi(\mathbf{q})\frac{\partial\phi(\mathbf{q})}{\partial\mathbf{q}} \qquad \{\mathbf{P}(\mathbf{q}),\phi(\mathbf{q})\} = \frac{\partial\phi(\mathbf{q})}{\partial\mathbf{q}}$$

Important for spatial correlations.



From the spatial derivative it is immediately clear in which directions the fields "move"

Digression

Another type of variational principle:

On a domain Ω in $\mathbb{R} \otimes \mathbb{R}^3$ (space(1,2,3)-time(0)):

$$0 = \nabla \frac{\delta \mathcal{L}}{\delta \nabla \phi} - \frac{\delta \mathcal{L}}{\delta \phi} \qquad T^{\mu\nu}_{energy-momentum} = \partial_\mu \phi \frac{\delta \mathcal{L}}{\delta \partial_\nu \phi} - g^{\mu\nu} \mathcal{L}$$

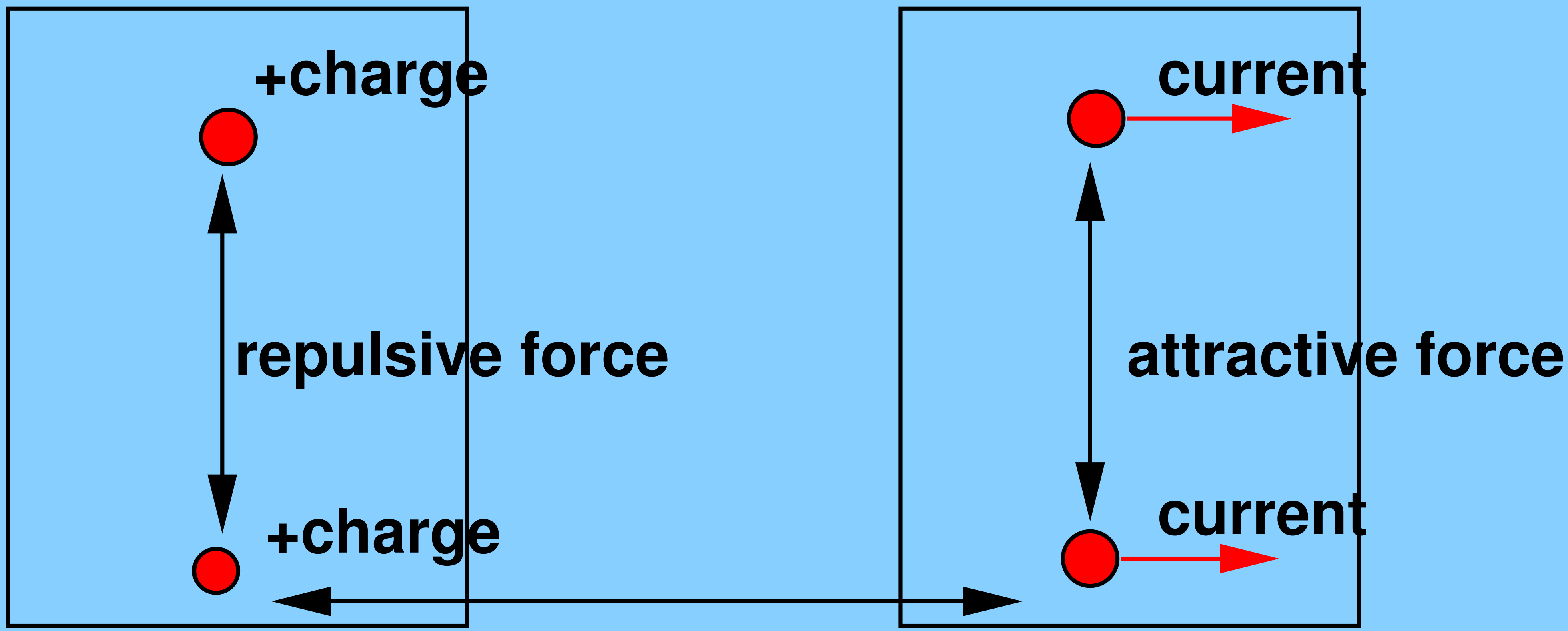
ϕ fixed on $\partial\Omega$.

$$\partial_\mu T^{\mu\nu} = 0 \qquad \text{and} \qquad P^\mu = T^{\mu 0}$$

Reduces for T^{00} to original Euler-Lagrange if:

$$\Omega = \{t \in [t_0, t_1] \wedge \mathbf{q} \in \mathbb{R}^3\}$$

electrostatic (Coulomb) magnetic (Faraday)



different inertial systems

$-d_tQ = \nabla \cdot J$ (Continuity relation, charge conservation)

Maxwell combined it: electromagnetism

wave–equation solutions, with invariant wave velocity

Gauge Fields A^μ are Tranverse Fields

(closed (but not exact since Bohm-Aharonov))

$$\mathbf{B}_{\text{magn}} = \nabla \times \mathbf{A} \quad \text{and} \quad \mathbf{E}_{\text{elec}} = -\nabla A^0 - \dot{\mathbf{A}}$$

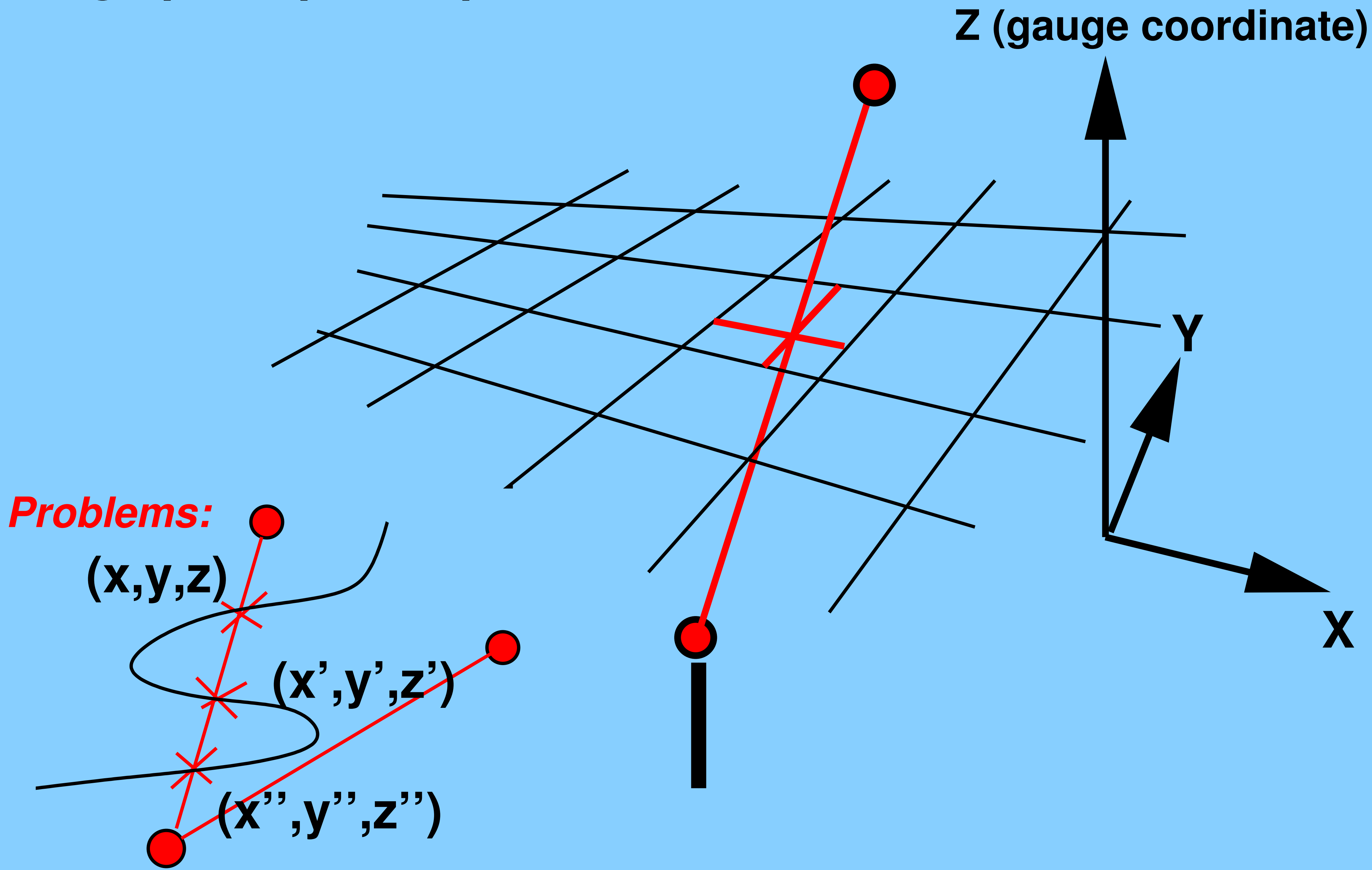
Hence invariant under $\mathbf{A} \to \mathbf{A} + \nabla\chi$ and $A^0 \to A^0 - \dot{\chi}$.

four-vector $s = (ct, x, y, z)$, four-current $j = (c\rho, j_x, j_y, j_z)$:

$\partial_\mu j^\mu = 0$, and $(\partial_t^2 - \nabla^2)A^\mu = j^\mu$.

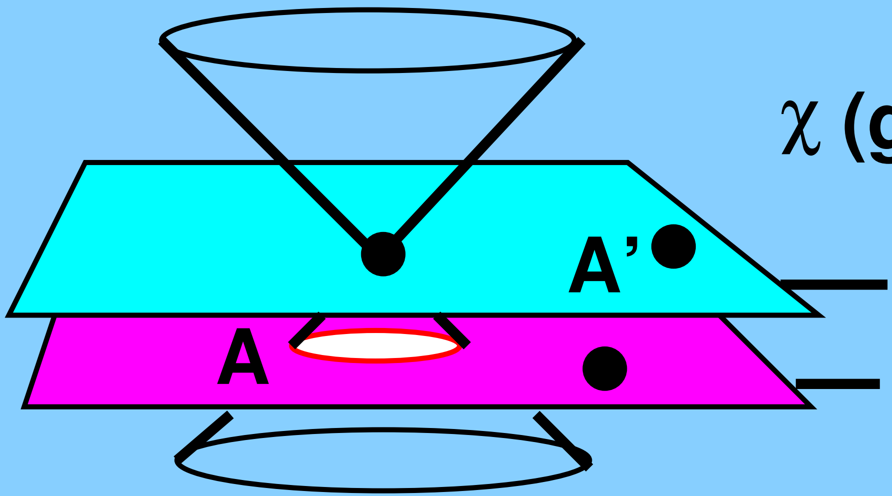
$$A^\mu(s) = \int d^4 s' \frac{j^\mu(s')}{|s' - s|^2} + \text{gauge terms}$$

Gauge principle: superfluous coordinates



Coulomb gauge: faster than the speed of light

light−cone (causal region)



χ (gauge) independent of time

$\nabla \circ A' = 0$ (at t')

$\nabla \circ A = 0$ (at t)

Continuity equation: charge conservation

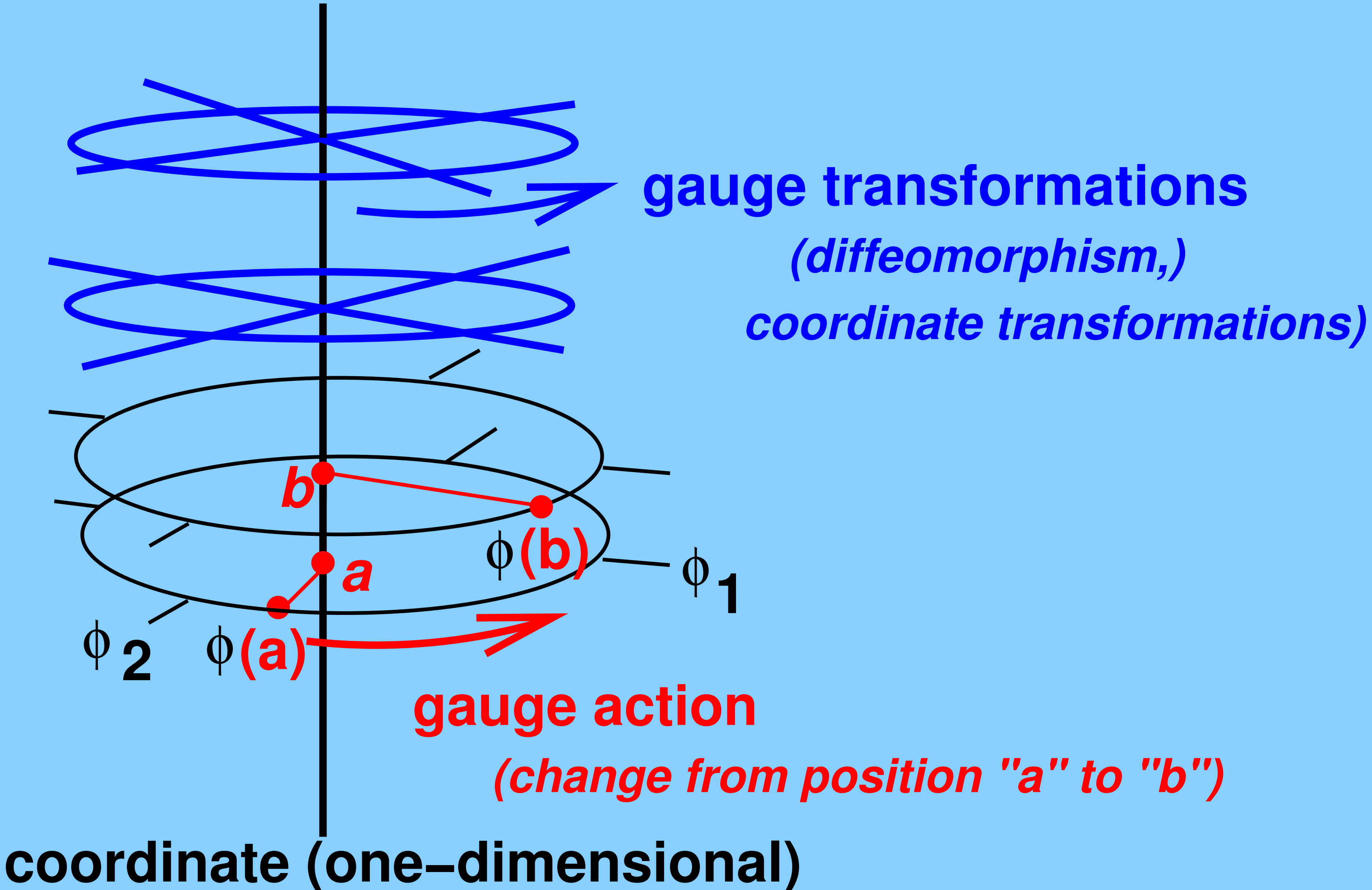
$A^{\mu=0\cdots3}$, a four-vector (Relativistic)

* Fields $\phi \rightarrow (\phi_1, \phi_2)$ (on a circle)

* Covariant derivative (Geometry):

$$\frac{\partial}{\partial x^\mu} \rightarrow D_\mu = \frac{\partial}{\partial x^\mu} - eA_\mu \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$*\mathcal{L}_{\text{gauge}} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad \text{where} \quad F_{\mu\nu} = [D_\mu, D_\nu]$$


gauge transformations
(diffeomorphism,)
coordinate transformations)
b
a
φ(b)
φ(a)
φ 1
φ 2
gauge action
(change from position "a" to "b")
coordinate (one–dimensional)

Digression

$$\mathtt{D}_\mu = \frac{\partial}{\partial x^\mu} - eA^a_\mu T^a \equiv \frac{\partial}{\partial x^\mu} - e\mathtt{A}_\mu$$

where T^a are Lie-group generators.

For particles that "belong" together: $\phi_1, \phi_2, \cdots, \phi_N$,

N-dimensional groups are used to postulate interactions.

$\Rightarrow$ non-linear dynamics, self-interactions (a mess):

$$F_{\mu\nu} = [\mathtt{D}_\mu, \mathtt{D}_\nu] = \partial_\mu \mathtt{A}_\nu - \partial_\nu \mathtt{A}_\mu + T^c f^{cab}_{\text{group}} A^a_\nu \cdot A^b_\mu$$

Nature $\equiv U(1) \otimes SU(2) \otimes SU(3)$ + gravity.

Eugene Wigner's legacy:

change of a system $\Rightarrow U$ where U unitary matrix : $\mathsf{U}^{\dagger} = \mathsf{U}^{-1}$

Conservation of probability: infinite state vector

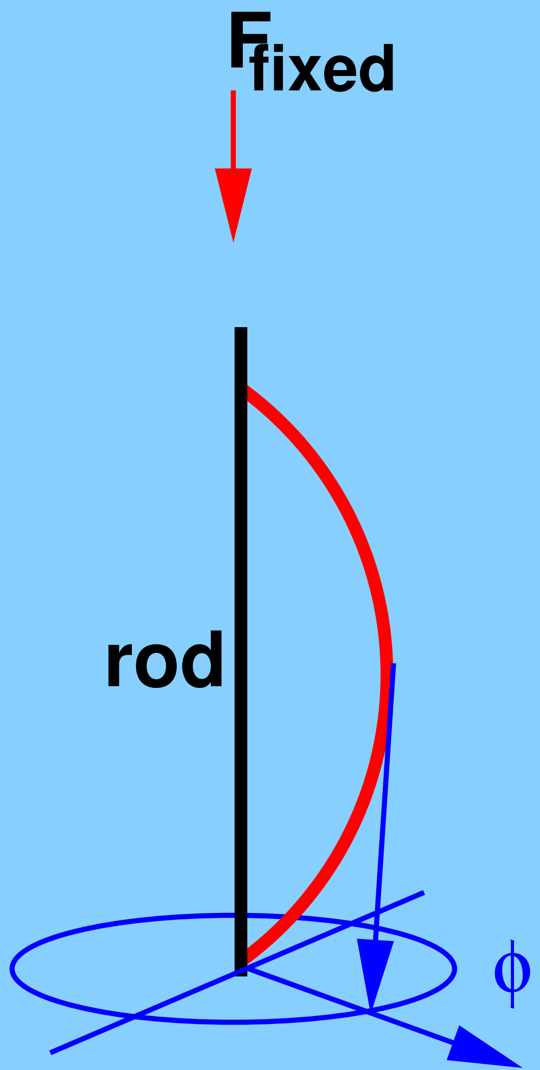
$\psi \in H_{\text{Hilbert}}$ over $\mathbb{C}$:

$$T_{\text{transform}}(\psi) = U \cdot \psi$$

such that $|\psi|^2 = 1.$

unitarity matrices rules!!

BUCKLING = SYMMETRY BREAKING

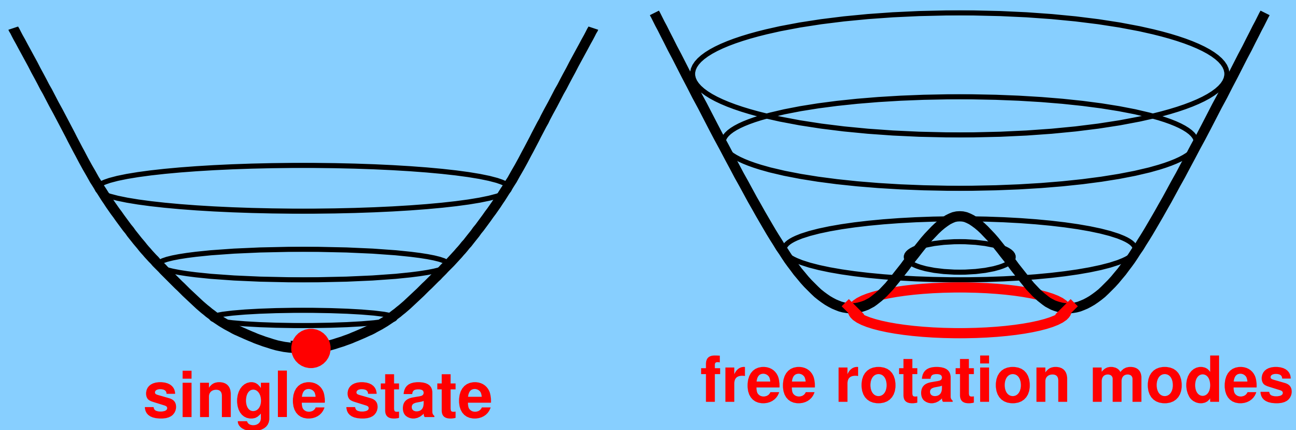


$$L(\phi) = L(\phi + \delta\phi)$$

Pick an angle ϕ_0

(The symmetry breaking state)

No force associated with a change of ϕ



Pions, the free modes of nuclear physics

particle triplet π^-, π^0, π^+, suprisingly light (139 MeV)

particle doublet p (proton), n (neutron), both 938 Mev,

where $p + \pi^- \to n$

$SU(2)_{\text{spin}} \otimes SU(2)_{\text{iso}} \approx SO(4) \to SO(3) \quad U(1)$ is broken

A 4-D sphere, with one axis fixed, is a 3-D sphere:

(π^-, π^0, π^+).

FINAL REMARKS

* The outside world: Sources, Sinks, and Schwinger

* Algebraic theories: spectra from group characters

* QCD (strongly interacting SU(3)) still unsolved

* Dominating experiments (particle physicists only know plane waves)