FUNCTIONAL SYSTEM DYNAMICS

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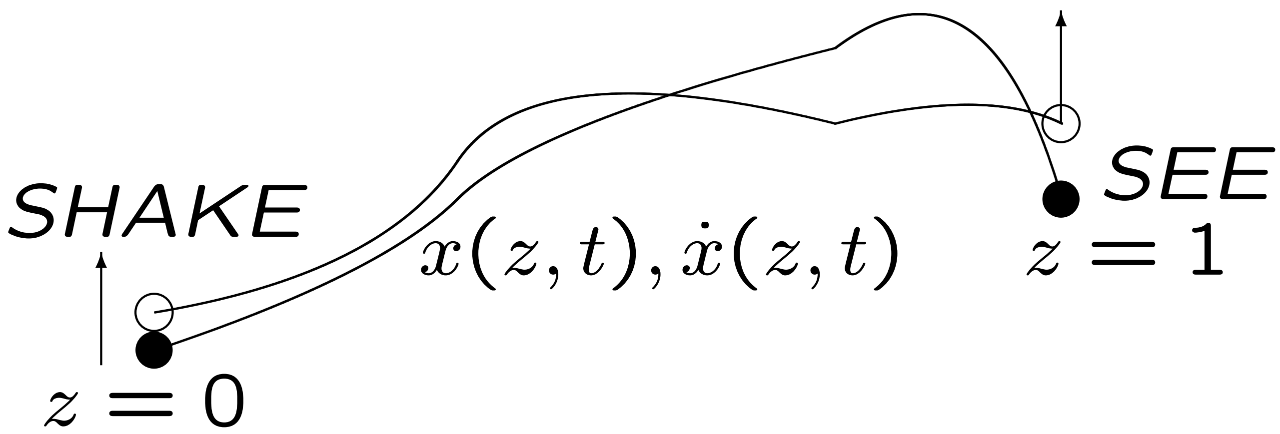
- Operators and functionals
- The limit to infinity
- Adding, removing, and replacing equations
- Boundary conditions, input-output
- More dimensions, Gauss, Green, and Stokes

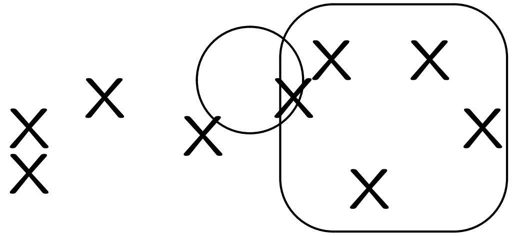
STATEMENT OF THE PROBLEM



Predict the Shake-and-See

seeking useful knowledge of the infinite world

TOPOLOGY or DISTANCE

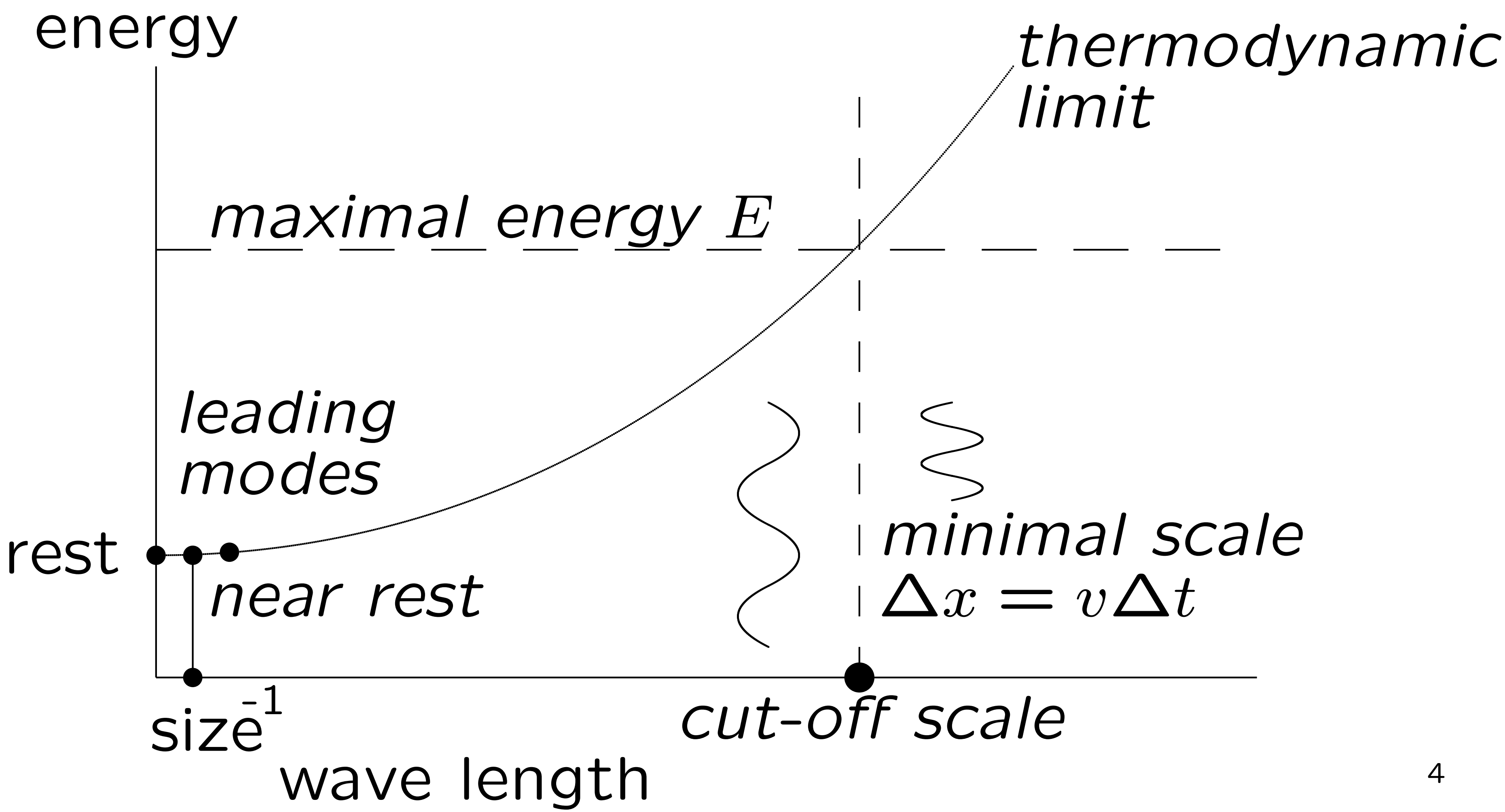


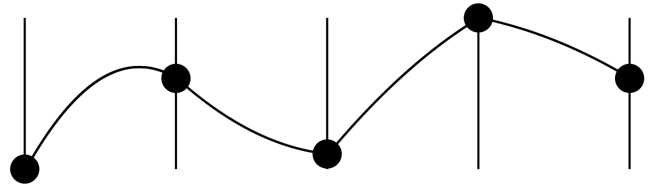
☐ The “index” $z \in \mathbb{R}$ of the degrees of freedom $x(z)$ is a mathematical point: unmeasurable.

☐ Without “differential operator” $\frac{\partial}{\partial z}$ a continuous model is just a collection of models $x(z) : \dot{x}(z) = A(x(z))$

☐ The differential operator introduces **ordering**, in the sense of *collective motion*, *correlation length* or *wave length*

DISPERSION is ordering



ABSCISSAS or SAMPLING 

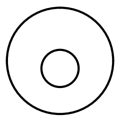
$$
\begin{aligned}
z &\longrightarrow \{z_0, z_1, \cdots, z_n\} \\
\frac{\partial}{\partial z} &\longrightarrow D :? \\
\int f(z)W(z)dz &\longrightarrow \sum_i w_i f(z_i) \\
e(z)f(z) &\longrightarrow e(z_i)f(z_i) \rightarrow \sum_{i,j} e(z_i)V_{ij}f(z_j)
\end{aligned}
$$

is only a restricted map of **functions** $\in C^0(\mathbb{R}) \rightarrow$ **vectors** $\in l_2$.

differential operator D (domain and range)

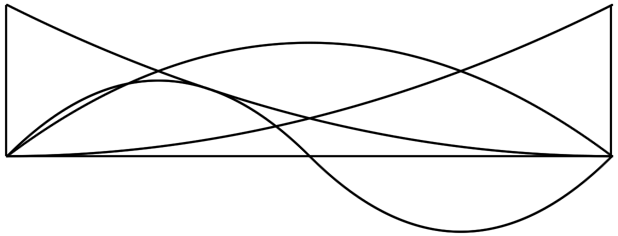
□ D is a linear, unbounded operator (e.g: $f(z) = z^{-0.3}$)

□ iD is only formally self-adjoint (even for the right, Liouville, boundary conditions)

□ Ker[D] is non-trivial (dependent on both function-space and topology) 

□ D needs to be defined (a space X of sampling points, yields an abstract $Y = DX$)

FUNCTIONS or MODES



$$f(z) \;\rightarrow\; \sum_i f_i \phi_i(z) = \mathbf{f}^T \phi$$

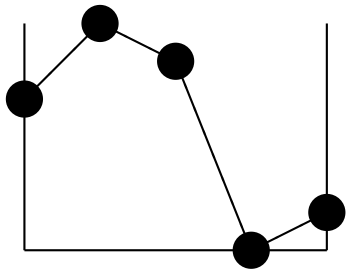
$$\frac{\partial}{\partial z} \;\rightarrow\; \partial_z \phi_i = \psi_i \rightarrow D : X \rightarrow Y$$

$$\int f(z) W(z) dz \;\rightarrow\; \mathbf{f}^T \mathbf{w}$$

$$e(z) f(z) \;\rightarrow\; \int e(z) f(z) dz \rightarrow \mathbf{e}^T \mathbf{V} \mathbf{f}$$

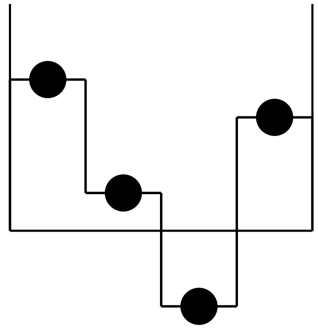
piece-wise
$\phi(z) = az + b$

piece-wise
$\psi(z) = a$



$\overset{D}{\longrightarrow}$

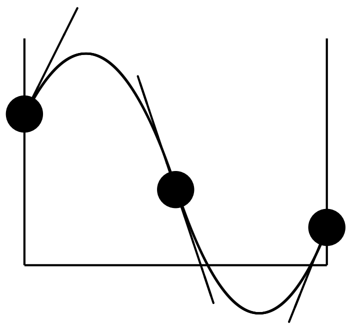
dim $X = 5$
dim $Y = 4$



Two pairs of local function spaces

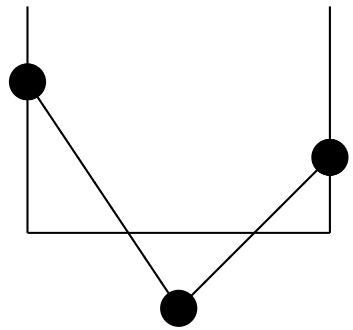
piece-wise
$\phi = az^2 + bz + c$

piece-wise
$\psi = 2az + b$

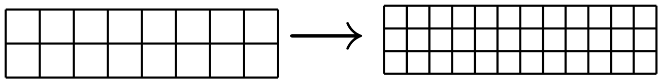


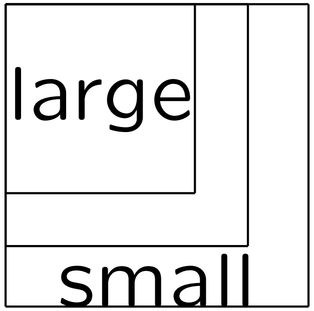
$\overset{D}{\longrightarrow}$

dim $X = 6$
dim $Y = 3$



THE POSSIBLE LIMITS TO ∞:

□ FEM 

□ Nuclear $J' =$ 

ordering modes

□ Green function expansion K^n

"transfer function" $K(x, y)$: x • → • y

Framework of function spaces, **functionals**:

$$\int e(f(z))f(z)w(z)dz \to J[f(z)]$$

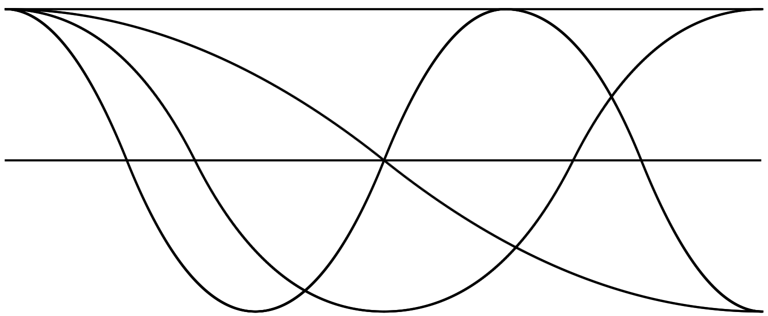
with their (useful) linear operators:

$$\lim_{\epsilon \to 0} \frac{J[f(z)+\epsilon u(z)] - J[f(z)]}{\epsilon} = J'[f(z)]u(z)$$

and variational principles: (min $J[u]$)

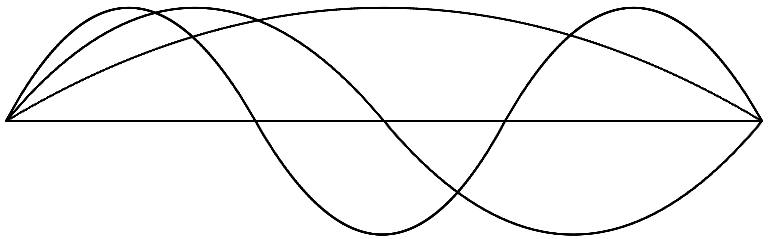
$$\delta J[u] = \delta \int (\nabla u)^2 + u^2 = 0 \quad \Leftrightarrow \quad \nabla^2 u = u$$

holonomic **CONSTRAINTS**
as restricted function spaces



Neumann boundaries
unrestricted (natural)

Dynamics is trivial



Dirichlet boundaries
restricted

Lagrange multipliers $\lambda(z)$ allows one the add or replace functions in the functional

$$J[u] \to J[u] + \lambda(z)G[u, v]$$

where $G[u, v] = 0$ is some constraint, or defines implicitly the functions v.

Lagrangian and **Hamiltonian** are special functionals associated with **dynamics**.

Lagrangian functional $L[u]$ with input force F

$$L[u] = \int \mathcal{L}(u, \dot{u}) + uF$$

The **Hamiltonian** arises from the elimination of velocity v, as an independent variable:

$$H[u, p] = p\dot{u} - L[u, \dot{u}]$$

where $p = \delta_{\dot{u}} L$ is the *Lagrange multiplier* of subsidiary condition $\dot{u} - v$.

time dependence: D'Alembert principle of the extremum of the action integral $S[u]$:

$$S[u] = \int_{t_0}^{t_1} dt L[u]$$

time-evolution of $u(t)$ is given by:

$$\delta S[u] = 0$$

More than one extremum might exists!

Removing functions $X \to X'$

The adiabatic, or instantaneous, approximation. Elasticity without inertia $M[\dot{u}] = 0$ follows the minimal energy $W[u]$ solution:

$$X' = \{u \in X \mid \delta W[u] = 0 \text{ and } M[\dot{u}] = 0\}$$

a massless spring will have homogeneous stress

(the problem will disappear when input-output systems; automatic function selection)

Typical quadratic potential terms in the ($z \in \mathbb{R}^1$) Lagrangian and Hamiltonian are (string, beam):

$$\left(\frac{\partial \phi}{\partial z}\right)^2 \;\rightarrow\; \phi^* D^* D \phi \tag{1}$$

$$\left(\frac{\partial^2 \phi}{\partial z^2}\right)^2 \;\rightarrow\; \phi^* D^* \overline{D}^* \overline{D} D \phi \tag{2}$$

spaces: $D : X \rightarrow Y$ and $\overline{D} : Y \rightarrow Z$. In nonlinear case: $\phi^* D^* D \phi \;\rightarrow\; f(\phi^*) D^* \Omega D \phi$. Sobolev type (co)energy: $E \approx \|\dot{u}\|_X + \|Du\|_Y$.

BOUNDARY CONDITIONS, INPUT-OUTPUT

questions of Functional System Dynamics

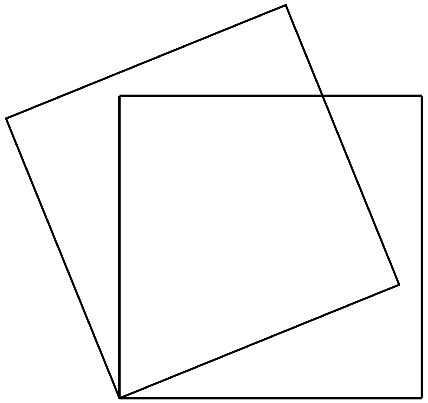
☐ To reduce the function-space of internal dynamics

☐ To choose the proper variables

☐ To relate global and microscopic properties

☐ To generate conservation-law interconnection structures

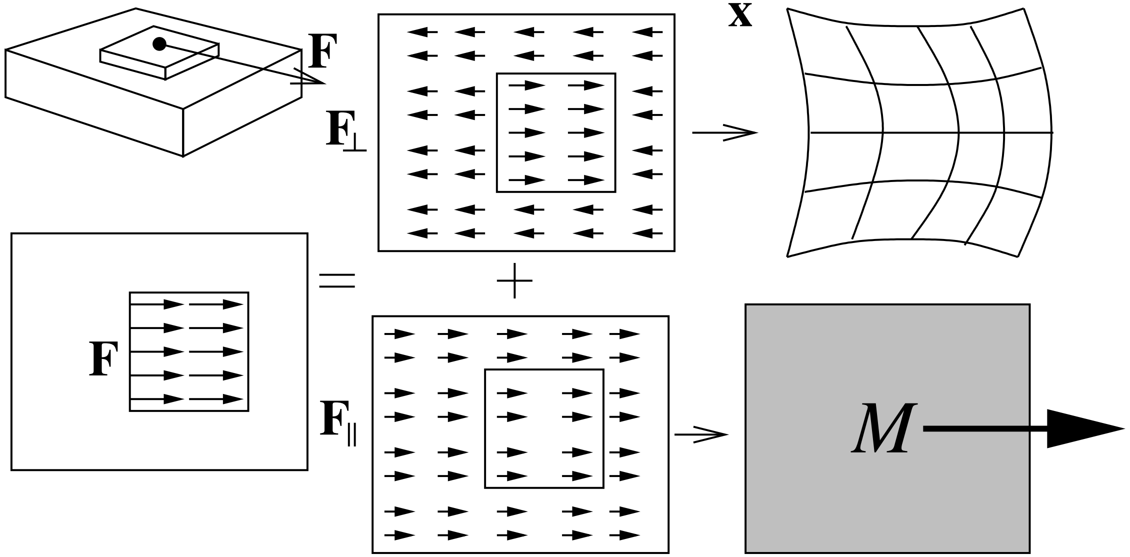
ELASTICITY internal *stress-strain*
external *force-displacement*



Rotation: displacement u_i
$\rightarrow$ but no strain ε_{ij}

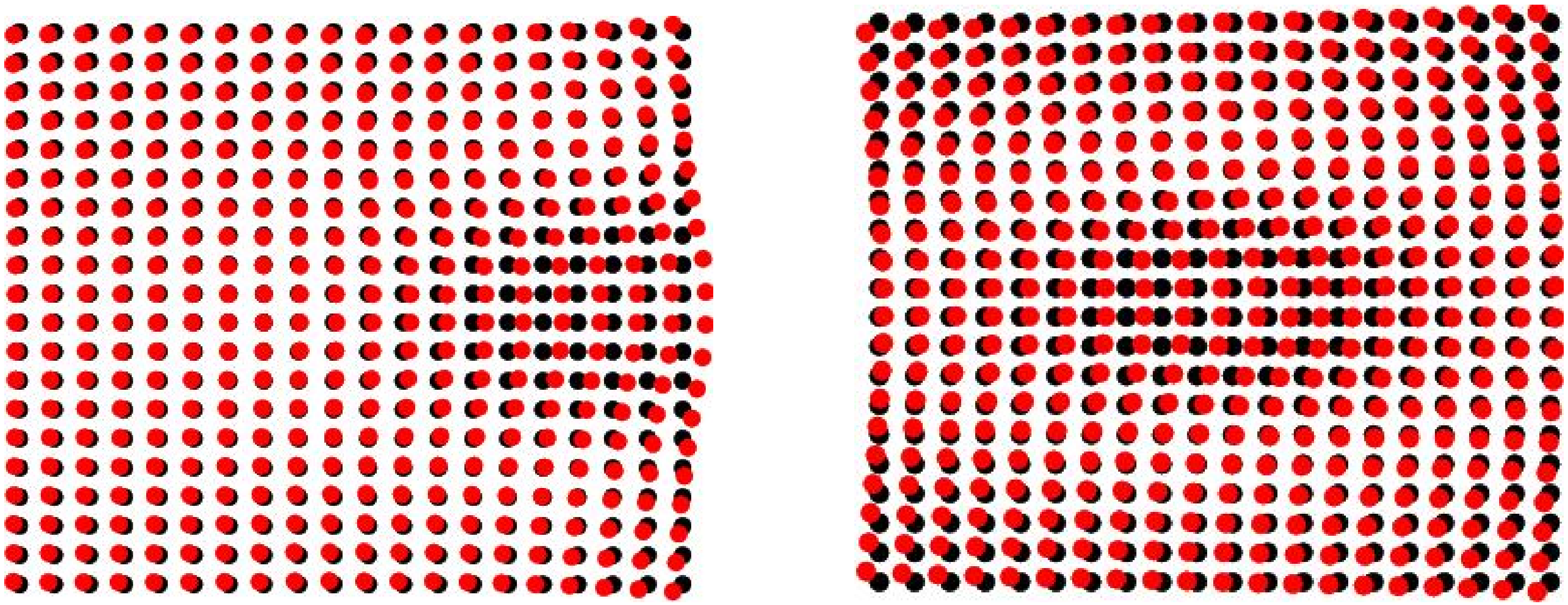
Inserting the relation between strain and displacement: $J[\varepsilon] \rightarrow J[\varepsilon] - \sigma_{ij}(\varepsilon_{ij} - \partial_i u_j - \partial_j u_i)$

However, $\nabla \times \mathbf{u} = 0$ would correspond to independence of χ in $\sigma \rightarrow \sigma + *\nabla\chi$ (problems!).



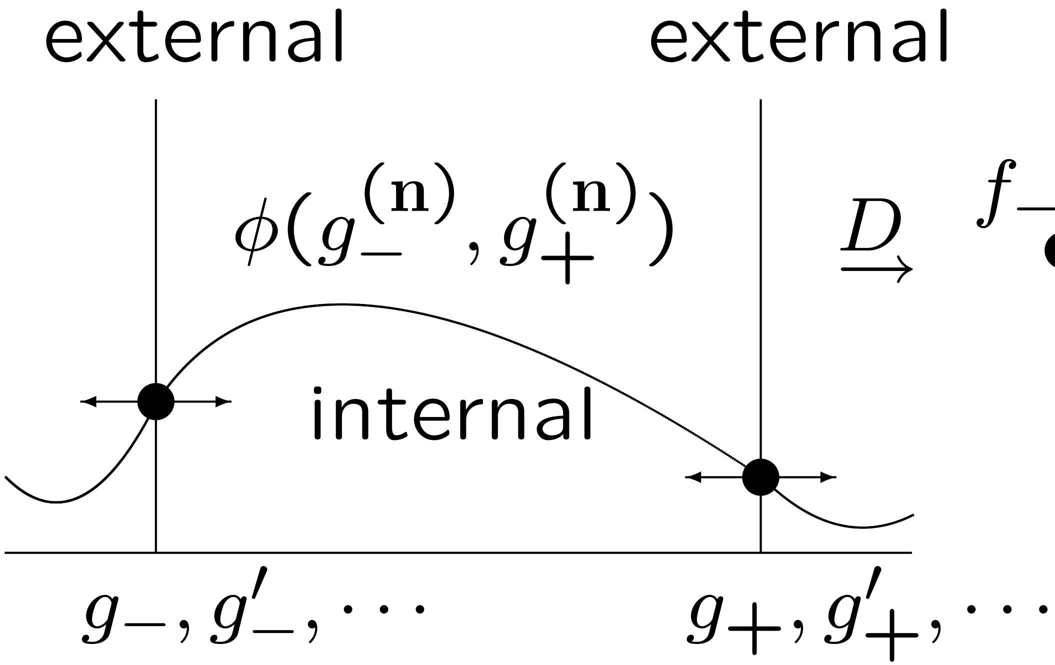
Only a limited number of functions are needed to describe the internal motion for a given applied force. However, for every fixture independently

AN EXAMPLE



dim $X = 800$, dim $Y = 2209$

Variables for internal and external use



Boundary polynomials

already incorporated in wave/diffusion dynamics code (Lyon collab.)

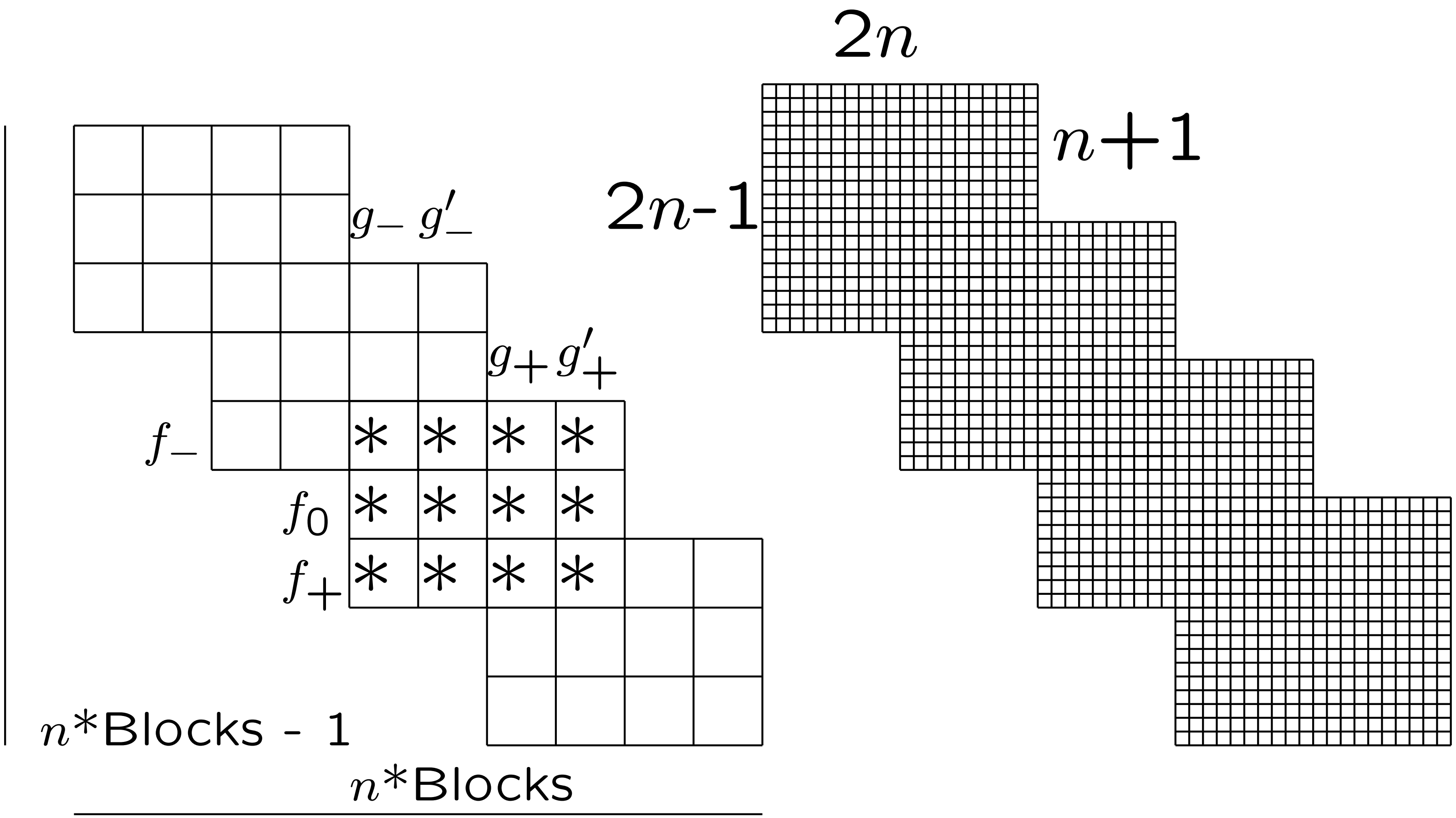
EXACT INTEGRATION ON $X \otimes Y$

The *exact* integral, in the sense of D^{-1}:

$$\int \psi_i(z)\phi_j(z)dz \to V_{ij}$$

with $(V : X \to Y)$, yields the power relation *rate-of-change = inflow - outflow*:

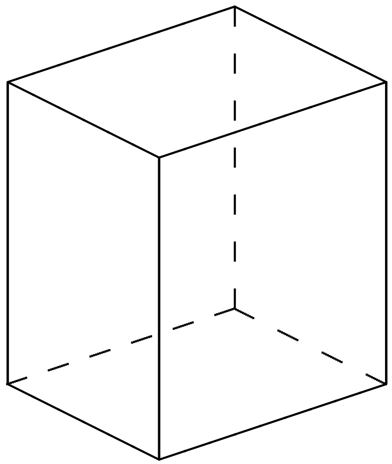
$$D^*V + V^*D = \delta_{++} - \delta_{--}$$



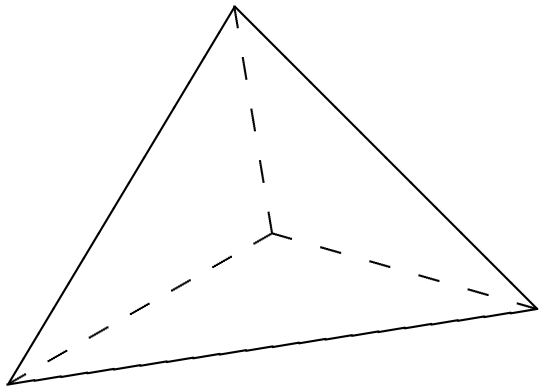
$$\dim X = \dim Y + 1$$

for any n and any number of segments

MORE DIMENSIONS, MORE D's (∇, $\nabla\times$, $\nabla\cdot$)



6 faces, 12 edges, 8 nodes



3 faces, 6 edges, 4 nodes

$\int \nabla \cdot E = E$(faces)

$\oint \nabla \times B = J$(edges)

$\int E = E$(volume)

$\dot{Q} = \nabla \cdot J \Rightarrow \sum J$(edges)$= \dot{Q}$(node)

exact (invertible) relations between integral quantities

WISH LIST

☐ A collection of $\{X, Y\}$'s for several D operators

☐ Also on curved spaces $z \in \mathcal{M}$, possibly in polynomial approximations ($z(s) = a + bs + cs^2$)

☐ Positivity criteria for D^*D operators (Hodge? Rham?)

☐ Systematic added, removing, and replacing equations

☐ Polynomial approximations of nonlinear $e(f)$ and $f(e)$

☐ Automation, software implementation