Physical model reduction of interacting, continuous systems

N. E. Ligterink, P. C. Breedveld, and A. J. van der Schaft

***Abstract*— For continuous physical systems described by partial differential equations, the energy density and the energy flux yield proper boundary conditions, which can serve as input and output. The minimal model is the energy flux solution depending solely on the boundary input, without any internal dynamics. The internal dynamics, given by a state vector, arises through an inverse expansion of modes around the minimal model state. Such an expansion yield a fast convergence of dynamics driven by the boundary input.**

I. Introduction

The analysis and simulation of systems described by Partial Differential Equations (PDE's) is an open problem. In many cases a linearized Finite Element Method (FEM) model suffices, if the corresponding A, B, C, D system is well-defined. However, the state x of a FEM model is usually too large for successive iterations needed for simulation and optimization. Hence, further model reduction is required. However, it is worthwhile to have a closer look at the original model in terms of PDE's and its qualities, in order to retain relevant features of the system at hand. The different relevant features of a system are not always intrinsic qualities of the model, in many cases, they depend on the setting, the operational conditions, and the interaction within the system network.

In the theory of model order reduction of linear systems, the features of the system are recognized as the structure of the respective matrices [2], such as the mass and stiffness matrices M and K. An appropriate model order reduction leaves these matrices symmetric and positive, which is not guaranteed by all methods. The systems under consideration are usually monolithic. A single state vector x incorporates all of the dynamics. If the system was composed of smaller components, this information was lost in the construction of the large system matrices.

For systems composed of smaller modular components, the interaction among them is a central feature, in particular to obtain physical insight. Since without interaction, each component will be isolated, and its dynamics often trivial. Even if systems interact, not all possible internal dynamics will be excited, only the modes driven by the input through other components are relevant, which should be retained in the reduced model.

In the mathematical literature of PDE's problems with both internal dynamics and input and output are mixed boundary value problems [5], or initial boundary-value problems [4]. In the case of nonlinear PDE's the existence and uniqueness is an open problem. For linear PDE's the Lopatinski condition [8] guarantees existence and uniqueness for elliptic PDE's. However, this condition can only be extended, as far as we know, to dynamical systems, described by hyperbolic PDE's if the PDE factorizes into characteristics [3]:

$$P(\partial_t, \nabla, z) = f(z) \prod_i (\partial_t + \mathbf{g}_i(z) \cdot \nabla) \quad ,$$

where $g_i(z)$ is the velocity field; the solution of the characteristic. The linear homogeneous wave equation in one-dimension has the well-known decomposition:

$$P = \partial_t^2 - c^2 \partial_z^2 = (\partial_t - c\partial_z)(\partial_t + c\partial_z) \quad ,$$

such that any solution to the wave equation is the sum of two arbitrary functions: $f(ct - z)$ and $g(ct + z)$. A boundary condition $u(t)$ at $z = 0$ for a system $z \geq 0$, would allow one to match the right-moving solution $f(ct - z)$ to the boundary condition minus the left-moving solution $g(ct + z)$. In many cases, such as higher dimensions and for realistic systems, constructing the characteristic is not a feasible approach. The Green function method incorporates these principles [6].

Furthermore, in the case of physical systems, we are not so much interested in some boundary condition of some PDE. The appropriate boundary conditions should follow from the appropriate PDE, which is describing the essential dynamics of the system. The energy flux plays an essential role in determining the appropriate boundary conditions.

II. Energy flux

The energy of the system is given by the Hamiltonian density function: $H(q, p)$, which depends on the canonical position and momentum fields $q(z)$ and $p(z)$. The energy is conserved, hence the Hamiltonian density satisfies the continuity equation:

$$\dot{H} + \nabla \cdot \mathbf{S} = 0 \quad ,$$

where $\mathbf{S}$ is the energy flux [1]. The energy flux is the key guiding principle is the analysis in the interaction between components. The energy function is a positive operator of the state space, or phase space, and zero energy corresponds to the lowest, rest, or ground state. Hence stability analysis and dissipation uses energy and its in- and out-flow.

The energy flux is not uniquely defined. It depends on what quantities are allowed to flow across boundaries. For example, an ideal balloon might change is shape but not its content, while an air pocket can both change shape and content. Constraints on the fields might complicate this

Norbert Ligterink and Peter Breedveld are members of the Control Engineering Group at the University Twente, Enschede, The Netherlands, Arjan van der Schaft is professor in Applied Analysis at the Mathematics Department of the University of Groningen, Groningen, The Netherlands, and professor in System and Control Theory at the University Twente.

matter further. The port-Hamiltonian H' defines the energy-flux. The port-Hamiltonian [9] arises from the Hamiltonian through a variable substitution:

$$H(q,p) = H'(Dq,p) \equiv H'(q',p) \quad ,$$

where D is an appropriate differential operator. The Hamilton-Jacobi equations, the equations of motion, are

$$\begin{pmatrix} \dot{q}' \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & D \\ -D^* & 0 \end{pmatrix} \begin{pmatrix} \delta_{q'} H' \\ \delta_p H' \end{pmatrix} \quad ,$$

where D^* is the formal adjoint of the differential operator D.

The energy flux follows from inserting the equations of motion into the time derivative of the Hamiltonian density:

$$\dot{H} = \delta_{q'} H' D \delta_p H' - \delta_p H' D^* \delta_{q'} H' \equiv -\nabla \cdot \mathbf{S} \quad .$$

The expression on the right-hand side is the divergence of the energy flux, which in many cases can be written as the bilinear product using the boundary operator θ:

$$\delta_{q'} H' \theta(\mathbf{n}) \delta_p H' = \mathbf{S} \quad .$$

Clearly, appropriate boundary conditions are expressed in terms of $\delta_p H'$ and $\delta_{q'} H'$ rather than the normal state variables q and p.

For the linear wave equation the Hamiltonian density is:

$$H = \frac{p(z)^2}{2\rho(z)} + \frac{1}{2}\kappa(z)(\nabla q(z))^2 \quad .$$

The associated differential operator is the gradient $D = \nabla$, the adjoint is the divergence $D^* = -\nabla\cdot$. The directional variable $q' = \nabla q$. Therefore the boundary operator θ is simply the surface normal:

$$\theta = \mathbf{n}$$

and the energy flux follows from the Green identity:

$$\mathbf{S} = \frac{p}{\rho}\mathbf{n} \cdot \kappa \nabla q \quad .$$

More general differential operators appear if there are more that one type of energy flux. For example, take the Euler-Bernoulli beam. The elastic energy is given by the second derivative of the deflection $y(z)$:

$$H_{\text{elastic}} = \frac{1}{2} EI (\partial_z^2 y(z))^2 \quad ,$$

which may seem like a single bending energy. The associated differential operator would be $D = \partial_z^2$ such that $q' = \partial_z^2 y$, however, the boundary operator θ is:

$$f\theta g = g\partial_z f - f\partial_z g \quad .$$

These two terms correspond to the two boundary conditions, which need to be set for an Euler-Bernoulli beam, at each end, which lead to three possible solutions for $\mathbf{S} = 0$: free ($\delta_{q'} H = \partial_z \delta_{q'} H' = 0$), clamped ($\delta_p H' = \partial_z \delta_p H' = 0$), and supported ($\delta_p H' = \delta_{q'} H' = 0$), boundary conditions, where in all cases the initial condition $q = 0$, at the boundaries, is used.

III. Minimal model

The initial boundary-value problem, or mixed problem, corresponds to data at an initial time, and data at the boundaries for the time-interval under consideration. The boundary conditions are the input to each system. Even in the absence of internal dynamics, forces are "transmitted" from one end to the other. Otherwise the Newton's basic force balance cannot be preserved. The energy flux plays a key role in linking boundaries together. If there is internal dynamics, such as vibrations, an internal state x is required to vibrate, and these internal states require initial data. In the absence of an internal state, the boundary values still influence each other, since the values at the boundary as a whole determine the static energy H'. The instantaneous, or adiabatic solution, which is the minimal energy solution for the given boundary conditions, we will call the minimal model of the system.

In different cases, the minimal model arises in different ways. For structural systems under force input one can think of the minimal model as a massless-spring model for internal deformations caused by opposite forces, together with a rigid-mass model for the acceleration due to the resultant, or sum, force. In the case of position input for a structural system, the free motion and the corresponding acceleration is absent, and the minimal model is the minimal elastic energy for given boundary conditions.

The minimal model (q_0', p_0) is the decomposition of a solution to an elliptic PDE. Furthermore, it is the lowest energy state for the given boundary conditions. If a spectral decomposition of the Hamiltonian is made, each vibration mode corresponds to a canonical pair of states $(q_i'(z), p_i(z))$, the remainder, is a function of the boundary only, and contains no dynamics at all. Hence, the state (q_0', p_0) corresponding to the minimal model, can be formulated as the solution to the variational problem:

$$\min_{(q',p)} \int dt \int dz H'(q',p) \quad ,$$

given:

$$B_q \delta_{q'} H' + B_p \delta_p H'|_\Gamma = u \quad ,$$

where B_q and B_p are the boundary conditions, in terms of the appropriate boundary variables; the variables of the energy flux.

Even is the canonical energy flux variables $\delta_{q'} H'$ and $\delta_p H'$ are used for the boundary conditions, finding the minimal model can be a nontrivial task. The minimal model is not always a constant solution, it can, as will be shown below, be an accelerating solution.

In the case of a string, the wave equation of before, in one dimension, $z \in [0,1]$ with a distributed mass $\rho(z)$ and varying elasticity $\kappa(z)$. See Figure 1. The total mass is:

$$M = \int_0^1 dz \rho(z) \quad .$$

If the outward forces at both ends are given F_0 and F_1. The minimal solution is the accelerating string with a constant

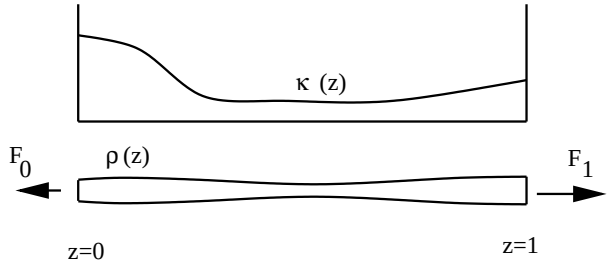

Fig. 1. A string with a nonhomogeneous mass $\rho(z)$ and elasticity $\kappa(z)$. The static boundary input are unequal forces F_0 and F_1 at either ends.

internal stress, which is the balance of external forces and inertial forces. The acceleration a, constant along the string, is the result of the resultant force:

$$Ma = F_1 - F_0 \quad ,$$

such that the force is constant in time: $D\delta_p H = 0$. The force, due to the inertia and the external force, along the string is given by the integral:

$$F(z) = F_0 + a\int_0^z dz\rho(z) \quad ,$$

such that the boundary condition matches: $F(1) = F_1$. The force is precisely the type of variable which is better expressed in terms of the variational derivatives $\delta_x H'$ than the state variables $x = (q', p)$. In this case, the force is directly related to the internal deformation:

$$F(z) = \delta_{q'(z)} H' = \kappa(z) q'(z) \quad ,$$

which, expressed in the boundary conditions is:

$$q_0'(z) = \frac{1}{\kappa(z)}\left(F_0 + \frac{F_1 - F_0}{M}\int_0^z dz\rho(z)\right) \quad ,$$

and the minimal model consists of a deformation which is a function of the force only, and an acceleration which depends on the time-integral of the resultant force and a single initial condition, or rest time t_0:

$$p_0(z,t) = \rho(z)\frac{F_1 - F_0}{M}(t - t_0) \quad .$$

The kinetic energy T is a quadratic function of time:

$$T(t) = \int dz \frac{p_0(z)^2}{2\rho(z)} = \frac{(F_1 - F_0)^2}{2M}(t - t_0)^2 \quad ,$$

which yield a minimum over a time domain, if t_0 is at its center.

The boundary condition in terms of the force F_0, F_1 is extended throughout the domain: $F(z)$ for $z \in [0, 1]$. The force is the solution to the differential equation:

$$\partial_z F(z) = a\rho(z) \quad ,$$

for an unknown acceleration a, given $F(0) = -F_0$ and $F(1) = F_1$. The minimal-model state variables follow directly from $F(z)$ and the equation of motion:

$$\dot{p}_0(z) = -D^* \delta_{q'(z)} H' = \partial_z F(z) \quad ,$$

where $D\delta_p H'(q', p_0) = 0$.

IV. Expansion around the minimal model

The effects of the boundary conditions are now incorporated in the minimal model. In the stationary case, with constant input, the minimal model is the full result. The variations in time of the input give rise to internal dynamics, such as vibrations. However, the internal dynamics is no longer a direct result of the boundary conditions but from the minimal model (q_0', p_0). The minimal model state serves as the background, or source, field for the internal dynamics. The resulting states and dynamics are automatically smooth, and the number of additional states required to reach convergence is small. The fact that the minimal model is the quasi-stationary result for the operational input guarantees the convergence, if the eigenfrequencies are typically higher than the time-variations of the input.

The expansion in terms of additional modes is generated by inserting the minimal model into the equations of motion. The constant input corresponds to the minimal model, the internal states are therefore generated through the expansion of the force:

$$F(t) = F + F^{(1)}t + \frac{1}{2!}F^{(2)}t^2 + \frac{1}{3!}F^{(3)}t^3 + \cdots$$

The minimal model $(q_0'(F(t)), p_0(F(t))$ tracks this expansion instantaneously, the deviations of the true dynamics are additional states:

$$\begin{pmatrix} q'(z,t) \\ p(z,t) \end{pmatrix} = \begin{pmatrix} q_0'(F(t)) + q_1'(q_0', p_0) + \cdots \\ p_0(F(t)) + p_1(q_0', p_0) + \cdots \end{pmatrix} \quad .$$

Hence, the equations of motion are not used to determine the time-derivatives $\dot{x}$, given the state x, but in reverse: given the time-derivative $\dot{x}_0 = (\dot{q}_0', \dot{p}_0) = (q_0'(\dot{F}), p_0(\dot{F}))$, which (q_0', p_0) is a solution to the stationary equations, to determine the corresponding change in states:

$$\begin{pmatrix} \dot{q}_0' \\ \dot{p}_0 \end{pmatrix} = \begin{pmatrix} 0 & D \\ -D^* & 0 \end{pmatrix} \begin{pmatrix} \delta_{q'} H'(q_0' + q_1, p_0 + p_1) \\ \delta_p H'(q_0' + q_1, p_0 + p_1) \end{pmatrix} \quad .$$

Since $(D^*\delta_{q'} H'(q_0', p_0), D\delta_p H'(q_0', p_0))$ is already a solution to the stationary equation, it might suffice to use a linear expansion of H' around (q_0', p_0), for the higher order terms in a nonlinear port-Hamiltonian.

However, several approaches and approximations exist to recover the next terms (q_1', p_1), (q_2', p_2), and so on, in the expansion. In the linear structural FEM case, the expansion reduces to the Krylov [2], [7] expansion $\mathcal{K}_n(A = K^{-1}M, B = (q_0', p_0))$, where M and K are the mass and stiffness matrices, and the starting state is the minimal-model state.

The additional work put in the generation of the modes (q_i', p_i) as the inverse expansion of the D, D^* operators will pay off in the actual simulation using the reduced model. The inverses of D and D^* are nonlocal operators and span the full domain. It would not be an appropriate expansion for very high orders, and very high accuracy, since the the corresponding eigenvalue problem is a Hankel, or moment, problem, known to yield oscillatory results in the order, after the initial fast convergence.

V. Conclusions

We have treated the initial boundary-value problem from the perspective of the physical system. The energy flux determines appropriate input and output variables. The minimal model is the lowest order model which can be devised for a particular choice of boundary conditions, or boundary input. It could be seen as the continuous analogue of the lumped model, of a component. Since the starting point of the analysis is the input of the system, in terms of the quantifier energy, it can serve to recuperate the leading order internal dynamics, due to this input. The expansion in the modes of the internal dynamics can be seen as the inverse expansion as used for time derivatives. The next order term counteracts the time-dependence of the previous term in the expansion.

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