Internal dynamics around static-deformation FEM states

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Abstract

Constant velocity or constant force FEM solutions are static-deformation states, where the elastic deformation is stationary. These are the typical operation conditions. Time-dependence, or fluctuations, of the static-deformation states are treated as perturbations, leading to a fast-converging expansion, for typical operation conditions, in which the time-scale of the input force is slower than the internal dynamics.

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1 Introduction

The premise that the design of a system is nearly ideal, and only small changes, or deviations, of the pre-described action occur, is as useful as it is realistic. To start the analysis from scratch, and recover the dynamics, and, on top, possible unwanted behavior is much more cumbersome, and requires much more numerical power, than to accept the given structure and seek out the corrections to the leading-order dynamics. The operational dynamics, or static-deformation dynamics, is the dynamics of the system as it is designed to operate. Possible deviations are usually considered as vibrations around this operational state. However, such vibrational analysis has several drawbacks. First, it is not immediately clear if and how vibrations are driven and how they depend on the different operation mode parameters, such as operation speed. Second, vibrational analysis is performed around a rest state, while in operation there might not be a clear rest state; the vibrational frequencies might vary along the operation trajectory. Third, the important effects, such as deviations from the desired trajectory are usually combinations of static-deformation deformations and vibrations, which are only partly described by vibrational analysis. Finally, as we shall see, it is usually the changes in static deformation which drive the vibrational, hence there is where design analysis should start.

2 Static-deformation solutions

The operation, in the traditional language of mechanical system, consists of a certain applied force, which causes the system as a whole to move. The distributed mass, given by the mass matrix $\mathbf{M}$, and the distributed friction $\mathbf{D}$, resist motion, and the internal force balance cause the system to deform with a displacement $\mathbf{x}$, determined by the stiffness matrix $\mathbf{K}$. The resulting equation of motion is the dynamic response $\mathbf{x}(t)$ of the applied force $\mathbf{F}$:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{D}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F}$$

We assume the system to be free to move, such that the overall displacement $\mathbf{q}$ is nothing but a change of

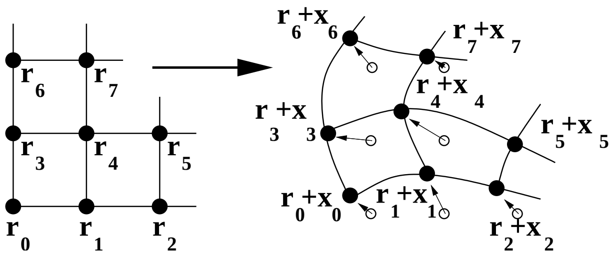

Figure 1: The reference coordinates $\mathbf{r}_i$ of the undeformed system, and the coordinates $\mathbf{r}_i + \mathbf{x}_i$, the sum of the reference coordinate and the displacements $\mathbf{x}_i$ of the deformed system. The elastic energy depends generally only on the displacements $\mathbf{x}_i$, however, the reference coordinates are important, e.g., in an accelerating frame.

frame, with no associated elastic force:

$$\mathbf{K}\mathbf{q} = 0 \quad (1)$$

Throughout the paper, different approximations of $\mathbf{x}$ are made. Most sections start with a particular choice of $\mathbf{x}$. Generally, they consists of two types of approximations. First, separating the time-dependent and the stationary part, where the time-dependent part corresponds to the rigid, or operational, motion. Second, constructing a number of modes and restricting the dynamics to these modes.

In most cases only the displacement $\mathbf{x}$ is of interest. However, in some cases, such as accelerating coordinate systems, or rotations, also the reference positions $\mathbf{r}$ are important, as we will see:

$$\mathbf{x} \rightarrow \mathbf{r} + \mathbf{x}$$

See Figure 1.

In the case of three spatial dimensions, the vectors $\mathbf{x}$ and $\mathbf{F}$ have $3N$ dimensions, for N points, or nodes:

$$\mathbf{x} = \left(\begin{pmatrix} x_1^x \\ x_1^y \\ x_1^z \end{pmatrix}, \begin{pmatrix} x_2^x \\ x_2^y \\ x_2^z \end{pmatrix}, \cdots \begin{pmatrix} x_N^x \\ x_N^y \\ x_N^z \end{pmatrix} \right)$$

$$= (x_1^x, x_1^y, x_1^z, x_2^x, x_2^y, x_2^z \cdots x_N^x, x_N^y, x_N^z)$$

The sum of vectors sum[$\mathbf{x}$] is therefore:

$$\mathrm{sum}[\mathbf{x}] = \begin{pmatrix} \sum_i x_i^x \\ \sum_i x_i^y \\ \sum_i x_i^z \end{pmatrix}$$

This sum selects the rigid-body dynamics of the systems; it can also be generated by projecting on all the rigid-body displacements $\mathbf{q}_i$, corresponding to translations of the system as a whole. Furthermore, vector operations, such as the outer-product, are assumed to be distributed:

$$\mathrm{sum}[\mathbf{x}\times\mathbf{F}] = \mathrm{sum}\left[(\mathbf{x}_1 \times \mathbf{F}_1, \mathbf{x}_2 \times \mathbf{F}_2, \cdots, \mathbf{x}_N \times \mathbf{F}_N)\right]$$

which, in this case, yields the sum torque.

2.1 Coordinate frames

The rigid-body modes Ker[$\mathbf{K}$] are a natural way to define an inertial system. For a free floating object, or free falling in the case gravity is taken into account, there are six rigid-body modes, corresponding to three translation directions, and three rotation directions. Hence a solution $\mathbf{q}$ of $\mathbf{K}\mathbf{q} = 0$ can be added at will to any solution. Furthermore, a constant solution $\mathbf{q}t$ for arbitrary time t can be added at will as well in the case of the translational modes, given by a linear subspace $\mathbf{K}\mathbf{q}t = 0$.

The internal configuration would naturally be the configurations $\mathbf{x}_r$ which are orthogonal to the rigid-body modes: $\mathbf{q}^T\mathbf{x}_r = 0$, or perpendicular to the null-space of $\mathbf{K}$. However, the orthogonality depends on the choice of metric. The traditional metric $\mathbf{q}^T\mathbf{x}_r$, based on the nodes, depends on the inhomogeneities of the grid. The common choice for internal configuration is the configuration that leaves the center of mass invariant: $\mathbf{M}\mathbf{x}_m = 0$, which is invariant for a change of coordinate system, for example, it is independent of inhomogeneities of the grid. It corresponds to the metric $\langle \mathbf{q}, \mathbf{x} \rangle_{\mathbf{M}} \equiv \mathbf{q}^T\mathbf{M}\mathbf{x}_m \rightarrow 0$. This configuration can be generated from the first internal configuration by adding a rigid translation, which is an element of the null-space of $\mathbf{K}$:

$$\mathbf{x}_m = \mathbf{x}_r - \sum_{i=x,y,z} \frac{\mathbf{q}_i^T\mathbf{M}\mathbf{x}_r}{\mathbf{q}_i^T\mathbf{M}\mathbf{q}_i}\mathbf{q}_i$$

where $\mathbf{q}_i$ is the rigid body translation in the i-th direction.

The details of the decomposition of the configuration in rigid motion and local deformation eventually boils down to the generalized inverse of the singular stiffness matrix $\mathbf{K}$.

Since the total, integral labor is the inner-product of force and displacement, the choice of metric for the displacement as an extensive quantity; weighted with the mass, yields an intensive force:

$$dE = dE_{\text{elastic}} + dE_{\text{rigid}} = \mathbf{F}_\perp^T d\mathbf{x} + \mathbf{F}_\parallel^T d\mathbf{q}$$

such that the decomposition of the force $\mathbf{F}$ into internal, or deformation, force $\mathbf{F}_\perp$ and global, or rigid-body, force $\mathbf{F}_\parallel$ is determined by: $\mathbf{F}_\perp^T \mathbf{q} = 0$ and $\mathbf{F}_\parallel^T \mathbf{x} = 0$. We will see, for example, that a matching acceleration $\mathbf{q}t^2$ for a particular rigid-body force $\mathbf{F}_\parallel$ yields a consistent decomposition, where the force can be interpreted as a null vector $\mathbf{q}_f$ of $\mathbf{K}$, weighted with the mass:

$$\mathbf{F}_\parallel = \mathbf{M}\mathbf{q}_f$$

which therefore satisfies $\mathbf{F}_\parallel^T \mathbf{x} = 0$. The physical interpretation is that each finite element with a given mass undergoes the same free acceleration through a force proportional to the mass. The force perpendicular to the rigid-body motion is the remainder of the total force:

$$\mathbf{F}_\parallel = \mathbf{M}\mathbf{q}_f \frac{\mathbf{q}_f^T \mathbf{F}}{\langle \mathbf{q}_f, \mathbf{q}_f \rangle_{\mathbf{M}}} \equiv \mathbf{F} - \mathbf{F}_\perp$$

Clearly, $\mathbf{q}_f^T \mathbf{F}_\perp = 0$. See Figure 2.

In the case of damping, the situation changes slightly, the resistance to motion is no longer the inertia, but a combination of resistive forces and inertia, the parallel component of the force $\mathbf{F}_\parallel$ is adapted to that situation. Furthermore, the inner-product $\langle \cdot, \cdot \rangle_*$ should also change. In this paper, the starting point is the equation of motion and the decomposition of the motion into a rigid-body motion, lying in the null-space of the stiffness matrix, and the static deformation. Rather than recovering the metric, we will project the equations on appropriate vectors, in the standard Hilbert-Courant l_2-sense.

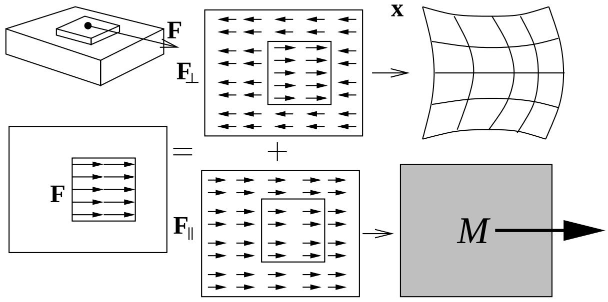

Figure 2: A simple example of the separation between rigid-body motion and its associated force and the remainder of the force causing the deformation. A block is pushed forward from the center. The rigid-body force $\mathbf{F}_\parallel$, calculated through sum$\mathbf{F}$, is distributed throughout the block, while the difference between the applied force and the inertial reaction force $\mathbf{F}_\perp = \mathbf{F} - \mathbf{M}\mathbf{a}$ causes the deformation $\mathbf{x}$. The rigid-body force $\mathbf{F}_\parallel$ causes the rigid-body motion on the lumped mass $M = \text{sum}[\text{diag}\mathbf{M}]$, without causing internal strain.

2.2 Constant acceleration

One of the most common operational mode, or static-deformation solution, is the case of constant force, causing a constant acceleration. We neglect the friction $\mathbf{D} = 0$. The displacement $\mathbf{x}(t)$ now consists of two terms: a constant deformation $\mathbf{x}$, which does not depend on the time t, and a global, or rigid, acceleration:

$$\mathbf{x}(t) = \mathbf{x} + \frac{1}{2}\mathbf{a}t^2$$

The acceleration $\mathbf{a}$ should match the applied force, which is the Newtonian rigid-body relation between the total mass, and the total force.

$$\text{sum}[\mathbf{F} - \mathbf{M}\mathbf{a}] = 0$$

which is a sum over N nodes only, yielding an equation for each spatial dimension. Furthermore, the lumped force should act on the center-of-mass, or the lumped mass, to prevent to object from rotating. Therefore the torque should also be zero:

$$\text{sum}[(\mathbf{r} + \mathbf{x}) \times \mathbf{F}] = 0$$

where $\mathbf{r} + \mathbf{x} = 0$ is the the center of mass, which depends on the deformation. Inserting this solution in the equation of motion yields:

$$\mathbf{Kx} = \mathbf{F} - \mathbf{Ma}$$

which can be satisfied despite the singularity in the stiffness matrix Eq. 1, since, $\mathbf{F} - \mathbf{Ma}$ is perpendicular to the null space of symmetric matrix $\mathbf{K}$. Due to the singular nature of $\mathbf{K}$ the solution $\mathbf{x}$ can be chosen such that $\text{sum}[\mathbf{Mx}] = 0$; the center-of-mass is independent of the deformation $\mathbf{x}$.

2.3 Constant velocity under friction

Another typical case is the maximal operating speed under constant force, caused by a balance between friction and force. In this case the damping $\mathbf{D}$ is not neglected. The solution is again constant deformation $\mathbf{x}$, and, in this case, a constant rigid-body velocity $\mathbf{v}$:

$$\mathbf{x}(t) = \mathbf{x} + \mathbf{v}t$$

Again the velocity should match the force, through the rigid-body relation:

$$\text{sum}[\mathbf{F} - \mathbf{Dv}] = 0$$

which yields the solution:

$$\mathbf{Kx} = \mathbf{F} - \mathbf{Dv}$$

Similar to the constant acceleration case, the structure deforms with $\mathbf{x}$, to distribute the force through the system. The two cases are specific examples of an augmented static force input F_i with zero mean, or rigid-body part, $\text{sum}[\mathbf{F}_i] = 0$:

$$\mathbf{Kx}_i = \mathbf{F}_i$$

The force is the sum of the applied force and the reaction force, either friction or inertia. Besides the condition of linear force balance, the torque balance is required to stop the object from spinning:

$$\text{sum}[\mathbf{x} \times \mathbf{F}_i] = 0$$

Otherwise the object would start spinning faster and faster around its center of mass. The $\text{sum}[\mathbf{F}_i] = 0$ and $\text{sum}[\mathbf{x} \times \mathbf{F}_i] = 0$ are not all the possible static-deformation solutions, in the case of rotation, an additional term arises, on the left-hand side of the equation.

2.4 Constant angular velocity

In the case of a rotating motion, around an axis, it is appropriate to choose the origin of the coordinate system such that is matches the axis: $\mathbf{x} \to \mathbf{x}_0 + \mathbf{x}$. Furthermore, the rest-position, or reference coordinate, $\mathbf{r}$ is relevant in this case. The static-deformation case is the constant angular velocity $\boldsymbol{\omega}$. The velocity of reference point $\mathbf{r}$ is given by:

$$\mathbf{v}(t) = \boldsymbol{\omega} \times (\mathbf{x} + \mathbf{r}) = \boldsymbol{\Omega}(\mathbf{x} + \mathbf{r}) \qquad (2)$$

which depends on the displacement, since if a point moves further away from the axis, it will move faster, for a constant angular velocity. Note, that we work in body-frame coordinates, which yields simpler expressions for the static deformation $\mathbf{x}$, cause by the centrifugal force. The time dependent position $\mathbf{x}(t)$ is given by the rotation matrix $\mathbf{R}$, which can be expressed in terms of the exponentiated generator of rotation $\boldsymbol{\Omega}$:

$$(\mathbf{x}(t) + \mathbf{r}(t)) = \mathbf{R}(t)(\mathbf{x} + \mathbf{r}) = e^{\boldsymbol{\Omega}t}(\mathbf{x} + \mathbf{r})$$

where the time dependence is solely in the rotation matrix:

$$\dot{\mathbf{R}}(t) = \boldsymbol{\Omega}\mathbf{R}(t)$$

Determining the acceleration:

$$(\ddot{\mathbf{x}} + \ddot{\mathbf{r}}) = \boldsymbol{\Omega}\boldsymbol{\Omega}(\mathbf{x} + \mathbf{r}) = -\Omega^2\mathbf{E}(\mathbf{x} + \mathbf{r})$$

where the $\mathbf{E}$ is the projection in the rotation plane, yielding the distance to the rotation axis $\boldsymbol{\omega}$: $\mathbf{E}^2 = \mathbf{E}$, $\boldsymbol{\omega}\mathbf{E} = 0$. For example, for the rotation around the z-axis, we find:

$$\boldsymbol{\Omega}_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{E}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Inserting the results in the equation of motion, for $\mathbf{D} = 0$:

$$(\mathbf{K} - \Omega^2\mathbf{ME})\mathbf{x} = \Omega^2\mathbf{MEr}$$

Generally, the second term on the left-hand side is small. In that case the the equation reduces to the generic case, however, without necessarily $\text{sum}[\Omega^2\mathbf{MEr}] = 0$ sum force zero, as the center of mass does not have to coincide with the rotation axis. It should be noted that applying the linear theory of elasticity to problems of spinning structure is dangerous, as for very large displacements instabilities occur.

2.5 Constant torque under friction

In the previous subsection we considered a constant angular velocity, without external torque, sustained through the lack of friction. Another static-deformation case of constant angular velocity would be the balance of friction with torque. This will lead only to a static-deformation state if the damping force is constant, which is, for example, the case if the system is invariant in the body-frame coordinates. The linear force, causing translation, is set to zero, as we only consider the angular motion:

$$\text{sum}[\mathbf{F}] = 0$$

while the applied torque balances the friction:

$$\text{sum}[(\mathbf{r}+\mathbf{x})\times\mathbf{F}] = \text{sum}[(\mathbf{r}+\mathbf{x})\times\mathbf{Dv}]$$

where $\mathbf{r} = \mathbf{x} = 0$ is the rotation axis. The angular velocity is the same as for the frictionless case Eq. 2. Note that this is a self-consistent problem; a change in deformation will cause both a change in friction and a change in applied torque, for a given force $\mathbf{F}$. However, the angular velocity Ω is easily adjusted to recover the static-deformation solution. Inserting this in the equation of motion, yields an augmented result, with the same radial equation as before, but an additional tangential equation:

$$(\mathbf{K} - \Omega^2\mathbf{ME})\mathbf{x} = \Omega^2\mathbf{MEr}$$

$$\mathbf{D\Omega x} = \mathbf{F} - \mathbf{D\Omega r}$$

where the applied force $\mathbf{F}$ is in the body frame, rotating with object. The two equations are coupled.

3 Slow changes

Static-deformation solutions can be combined to generate an approximate solution for a more general force input. The dynamics of the deformation is ignored. Given a general rigid-body motion, a combination of acceleration and velocity: $\frac{1}{2}\mathbf{a}t^2 + \mathbf{v}t \to \mathbf{q}(t)$, inserted in the equation of motion with $\alpha + \beta = 1$, such that the total force is $\mathbf{F}$, yields:

$$\mathbf{Kx} = \mathbf{F} - \mathbf{M}\ddot{\mathbf{q}} - \mathbf{D}\dot{\mathbf{q}} = (\alpha\mathbf{F} - \mathbf{M}\ddot{\mathbf{q}}) + (\beta\mathbf{F} - \mathbf{D}\dot{\mathbf{q}})$$

where the deformation is the corresponding linear combination of the static-deformation solutions for constant velocity for the force $\mathbf{F}$, and constant acceleration in the absence of friction for the same force:

$$\mathbf{x}(t) = \alpha(t)\mathbf{x_a} + \beta(t)\mathbf{x_v} + \mathbf{q}(t)$$

where $\mathbf{x_a}$ and $\mathbf{x_v}$ are the static-deformation constant acceleration and constant velocity solution, each for the force $\mathbf{F}$. The absence of deformation dynamics corresponds to ignoring the time dependence of the fractions: $\dot{\alpha} = \ddot{\alpha} = 0 = \dot{\beta} = \ddot{\beta}$. Neglecting the time-derivatives in the dynamics is sometimes referred to as the adiabatic approximation. We will keep referring to it as the instantaneous approximation.

The rigid-body equation is the standard damping equation, now microscopically motivated through the mass and damping matrices:

$$M\ddot{\alpha} + D\dot{\alpha} = 0$$

where $M = \text{sum}[\mathbf{Mv}]$ and $D = \text{sum}[\mathbf{Dv}]$.

The solution is an asymptotic approach of the constant velocity under friction limit.

$$\alpha(t) = \exp\left\{-\frac{D}{M}t\right\}$$

for a given force $\mathbf{F}$.

4 Perturbations around static-deformation solutions

In the case of control, one would like to vary the input force: $\mathbf{F}(t) = \alpha(t)\mathbf{F}$, and determine the effects of the deformation, rather than assuming the deformation varies instantaneously. In the simplest case one assumes the change of deformation to be parallel to the deformation itself. Given an instantaneous solution $\alpha(t)$, the direct result of the static-deformation response to the force $\mathbf{F}(t)$ at a certain time t, the first dynamical correction would be a small change $\beta(t)$ due to the change in deformation:

$$\mathbf{x}(t) = (\alpha(t) + \beta(t))\mathbf{x} + \mathbf{q}\int^{t} dt' \int^{t'} dt'' \alpha(t'')$$

where we assume that the correction is small compared to the original static-deformation deformation: $\alpha \gg \beta$

Inserting the restricted solution in the equation of motion, and projecting on the static-deformation deformation $\mathbf{x}^T$, which guarantees positive-definite values for mass and damping, to reduce the $3N$ equation to a single equation for the single unknown β, we find:

$$\mathbf{x}^T\mathbf{M}\mathbf{x}\ddot{\beta} + \mathbf{x}^T\mathbf{D}\mathbf{x}\dot{\beta} + \mathbf{x}^T\mathbf{K}\mathbf{x}\beta = -\mathbf{x}^T\mathbf{M}\mathbf{x}\ddot{\alpha} - \mathbf{x}^T\mathbf{D}\mathbf{x}\dot{\alpha}$$

Defining $m = \mathbf{x}^T\mathbf{M}\mathbf{x}$, and k and d likewise, the expression reduces to a simple forced and damped oscillator:

$$m\ddot{\beta} + d\dot{\beta} + k\beta = -m\ddot{\alpha} - d\dot{\alpha} \equiv f(t)$$

As an input-output system, with input α and output β, the transfer function H, such that $\beta = H\alpha$, is a rational function with two poles and two zeros:

$$H = -\frac{ms^2 + ds}{ms^2 + ds + k}$$

In the low-frequency regime, the leading dynamical effect is caused by the inertia, and yields a delayed deformation, generally proportional to m/k, if damping can be ignored.

More generally, one would like to consider dynamical deformation effects, where the deformation is not necessarily parallel to the static-deformation deformation:

$$\mathbf{x}(t) = \alpha(t)\mathbf{x} + \beta(t)\mathbf{y} + \mathbf{q}\int^{t} dt' \int^{t'} dt'' \alpha(t'')$$

In this case not only the amplitude of the dynamical deformation β, but also the shape $\mathbf{y}$, is unknown. However, they are assumed small, such that higher-order time derivatives of $\beta(t)$ can be ignored. Inserting this into the equation of motion:

$$\mathbf{M}\mathbf{y}\ddot{\beta} + \mathbf{D}\mathbf{y}\dot{\beta} + \mathbf{K}\mathbf{y}\beta = -\mathbf{M}\mathbf{x}\ddot{\alpha} - \mathbf{D}\mathbf{x}\dot{\alpha}$$

which yield for every instance in time a dynamic deformation:

$$\beta\mathbf{y} = -(\mathbf{M}\frac{\ddot{\beta}}{\beta} + \mathbf{D}\frac{\dot{\beta}}{\beta} + \mathbf{K})^{-1}(\mathbf{M}\ddot{\alpha} + \mathbf{D}\dot{\alpha})\mathbf{x}$$

A whole range of solutions is the result. For every combination ratios: $\ddot{\beta}/\beta : \dot{\beta}/\beta : \ddot{\alpha} : \dot{\alpha}$, a different deformation $\beta\mathbf{y}$ exists. For the purpose of generating the leading-order perturbation we assume damping and higher order-derivatives of β to be zero: $\mathbf{D} = 0$, and $\ddot{\beta} = \dot{\beta} = 0$, such that the dynamical deformation is proportional to:

$$\mathbf{y} \sim \mathbf{K}^{-1}\mathbf{M}\mathbf{x}$$

Inserting the assumed mode shape $\mathbf{y}$ in the equation of motion yields:

$$\mathbf{M}\mathbf{x}\left(\ddot{\alpha} + \beta\right) = -\mathbf{M}\mathbf{K}^{-1}\mathbf{M}\mathbf{x}\ddot{\beta} \approx 0$$

When the small term on the right-hand side is set to zero, the $3N$ equations for β reduce to $3N$ copies of the simple equation: $\ddot{\alpha} + \beta = 0$. Hence, the inertia of the static-deformation deformation $\mathbf{x}$, while changing, generates a force which is counter-acted by the a deformation $\mathbf{y}$. The fact that under the neglect of the time derivatives of β, the equation occurs in $3N$ copies, means it is a, in that approximation, exact result. Generally, one has to decided in what extent each of the $3N$ equations is satisfied, which will be discussed below in the section on self-consistent projection.

5 Expansion in perturbations around the static-deformation state

So far, we assumed perturbations around the static-deformation state as an instantaneous approximation to the time-dependent force to be small. However, it is possible to generate a systematic expansion around the static-deformation state, which can be truncated at the required accuracy. Still the number of terms in the expansion will be much less than the $3N$ terms of the full equation of motion. The general principle is to assume the deformation to be instantaneous for a particular set of modes. The time-dependence of the deformation gives rise to additional inertial effects, compensated by an additional deformation, which is the next mode in the expansion. The inertia, or dynamics, of a deformation mode are balanced by the forces of the additional elastic deformation.

For an arbitrary force input: $\mathbf{F}(t) = \alpha_0(t)\mathbf{F}$, the time dependence is given by α_0, while the spatial distribution is given by $\mathbf{F}$, with an arbitrary normalization. For the moment damping is not considered to influence the deformation greatly beyond the static-deformation result. Changes in force input are more likely to cause vibrations, for which the mass and the elasticity are important. We will, however, take into account the effect of damping on the mode shapes. The static-deformation solution for $\alpha_0(t) = 1$ is $\mathbf{q}(t) + \mathbf{x}_0$ where $\mathbf{q}(t)$ is the rigid-body motion, and $\mathbf{x}_0$ the static deformation. We expand the full solution:

$$\mathbf{x}(t) = \mathbf{q}(t) + \alpha_0(t)\mathbf{x}_0 + \alpha_1(t)\mathbf{x}_1 + \alpha_2(t)\mathbf{x}_2 + \alpha_3(t)\mathbf{x}_3 \cdots \quad (3)$$

where generally the coefficients decrease in order: $\alpha_0 \gg \alpha_1 \gg \alpha_2 \gg \cdots$. The mode shapes of the deformations $\mathbf{x}_n$ is given by the Krylov expansion:

$$\mathbf{x}_n = \left(\mathbf{K}^{-1}\mathbf{M}\right)^n \mathbf{x}_0$$

Since the stiffness matrix $\mathbf{K}$ is singular, the inverse should be considered the generalized inverse, with the same null space: $\mathrm{Ker}[\mathbf{K}] = \mathrm{Ker}[\mathbf{K}^{-1}]$. The mode shape $\mathbf{y}$ of the previous section corresponds now to the second term in the expansion $\mathbf{y} = \mathbf{x}_1$. Each of the mode shape is an independent degree of freedom. If the higher order time derivatives are ignored, the equation of motion yield a hierarchy of equations for the coefficients:

$$\ddot{\alpha}_n + \alpha_{n+1} = 0 \quad (4)$$

However, this will not yield the internal oscillations which are part of the deformation dynamics, which require the full equation of motion:

$$\sum_{n=0} \mathbf{M}\mathbf{x}_n(\ddot{\alpha}_n + \alpha_{n+1}) + \mathbf{D}\mathbf{x}_n\dot{\alpha}_n = 0 \quad (5)$$

where the static-deformation solution satisfies the instantaneous equation, where the time derivatives of α_0 are ignored:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{K}\mathbf{x}_0 = \mathbf{F}$$

One should reduce the $3N$ equations of motion to a number of equations equal to the number of coefficients α_n.

5.1 Self-consistent projection

One should distinguish two parts to a reduced dynamical system, such as the dynamics in the subspace $\{\mathbf{x}_n\}_{n=0}^{A-1}$. First, the construction of the modes $\mathbf{x}_n$, and, second, the dynamics itself. For the construction of the modes the damping is ignored, apart from the static-deformation part of the damping. However, for the dynamics of these modes the damping is taken into account. While constructing the modes, the true system is approximated, and one is accutely aware of the missing degrees of freedom. However, once the modes constructed, the reduced-model dynamics should be self-consistent, such that, e.g., the energy is conserved. Internal dynamics is mainly the results of driven oscillatory behavior; the interplay of potential and kinetic parts. The damping is expected to play only a role at the longer time scales, and therefore not determine the mode shapes. Hence, in the construction of the modes, damping effects can generally be ignored.

The $3N$ dimensional equations Eq. 5 can be reduced to a consistent set of equations for the A coefficients α_n by projecting on the dual basis $\mathbf{z}_i$ of $\mathbf{M}\mathbf{x}_j$, which satisfies:

$$\mathbf{z}_i^T \mathbf{M}\mathbf{x}_j = \delta_{ij}$$

which, in the absence of the damping term $\mathbf{D} = 0$, leads to the hierarchical equations Eq. 4. However, these simple equations have a sting in the tail, as they lead to numerical instabilities and inconsistent results. In order to understand this we should look in detail at the equation of motions and the limited freedom we allow for the deformation $\mathbf{x}(t)$ in Eq. 3. The equation of motion tells us the evolution of the system. Hence the $\ddot{\mathbf{x}}$ term in the equation of motion should lie in the subspace spanned by: $\{\mathbf{x}_n\}_{n=0}^{A-1}$.

Such a criterion is impossible to derive from the equations of motion, since generally the dynamics is not restricted to a subspace. Instead we investigate the generating functional of the equations of motion, and postulate a Lagrangian, for the dynamics of the α_i's, which generates the projected equations of motion and satisfies the positivity and symmetry criteria of dynamical systems:

$$L(\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}) = \frac{1}{2} \sum_{ij} \dot{\alpha}_i \mathbf{x}_i^T \mathbf{M} \mathbf{x}_j \dot{\alpha}_j - \alpha_i \mathbf{x}_i^T \mathbf{K} \mathbf{x}_j \alpha_j \quad (6)$$

The damping term is absent, as it is normally not part of an energy-conserving Lagrangian, however, it can be constructed along the same line. Clearly, the mass and stiffness parts are positive definite and symmetric matrices of dimension $A \times A$. The resulting equations of motion for each m are:

$$\sum_{n=0} \mathbf{x}_m^T \mathbf{M} \mathbf{x}_n (\ddot{\alpha}_n + \alpha_{n+1}) + \mathbf{x}_m^T \mathbf{D} \mathbf{x}_n \dot{\alpha}_n = 0$$

which is the result of the least-action principle, adding the positive definite damping by hand:

$$\frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{\alpha}_m} - \frac{\partial L}{\partial \alpha_m} = 0 \quad (7)$$

applied to the Lagrangian Eq. 6. In this case, oscillations of the deformation can occur, as can be seen from writing the equation of motion for α_i only, in its native form:

$$\mathbf{x}_i^T \mathbf{M} \mathbf{x}_i \ddot{\alpha}_i + \mathbf{x}_i^T \mathbf{D} \mathbf{x}_i \dot{\alpha}_i + \mathbf{x}_i^T \mathbf{K} \mathbf{x}_i \alpha_i =$$

$$m_i \ddot{\alpha}_i + d_i \dot{\alpha}_i + k_i \alpha_i = 0$$

where we set all other coefficients $\alpha_j = 0$, including their variations in the Euler-Lagrange equation Eq. 7.

6 Control

So far we have considered the deviations of the static-deformation state from the perspective of internal dynamics. However, it can also be viewed from the control perspective; how the internal dynamics depends on time dependence of the input force $\mathbf{F}(t)$. Force input is generally smooth, such that it can be approximated by an analytical function of the time t for a finite time:

$$\mathbf{F}(t) = \alpha_0(t)\mathbf{F} = (\alpha_{00} + \alpha_{01} t + \frac{1}{2!}\alpha_{02} t^2 + \cdots)\mathbf{F}$$

Instead of looking at the time evolution, we will investigate the analytical expansion around $t = 0$, which yields the same information. Inserting this into the equation of motion, differentiating n times, and setting $t = 0$:

$$\mathbf{M}\mathbf{x}^{(n+2)}(0) + \mathbf{D}\mathbf{x}^{(n+1)}(0) + \mathbf{K}\mathbf{x}^{(n)}(0) = \alpha_{0n}\mathbf{F} \quad (8)$$

Making a similar series expansion of the time dependence of the internal deformation:

$$\mathbf{x}(t) = \mathbf{q}_0 t + \frac{1}{2}\mathbf{q}_1 t^2 + \alpha_0(t)\bar{\mathbf{x}}_0 + \alpha_1(t)\bar{\mathbf{x}}_1 + \alpha_2(t)\bar{\mathbf{x}}_2 + \cdots$$

where $\mathbf{q}_1$ and $\mathbf{q}_0$ are the rigid-body acceleration and velocity of the static-deformation state, corresponding to $\alpha_{00}\mathbf{F}$, yielding the static deformation $\bar{\mathbf{x}}_0$. The coefficients $\alpha_n(t)$ are given by the integrated input:

$$\alpha_n(t) = \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \alpha_0(t_n)$$

such that the n-th derivative of the m-th term at $t = 0$ is ($n \geq m$):

$$\bar{\mathbf{x}}_m^{(n)}(0) = \alpha_{0(n-m)}\bar{\mathbf{x}}_m$$

where $\bar{\mathbf{x}}_m(t) = \alpha_m(t)\bar{\mathbf{x}}_m$; a single term in the expansion. The leading order term of each coefficient $\alpha_n(t)$ is $\alpha_{00}t^n/n!$. Inserting this into the equation of motion, at $t = 0$ reduces to the static-deformation problem. Inserting this expansion in the differentiated equations of motion Eq. 8, we find that in the absence of damping $\mathbf{D} = 0$, the expansion corresponds to the even terms of the Krylov expansion of the static-deformation state:

$$\alpha_{01}\mathbf{K}\bar{\mathbf{x}}_1 + \alpha_{01}\mathbf{K}\bar{\mathbf{x}}_0 = \alpha_{01}\mathbf{F}$$

$$\alpha_{02}\mathbf{M}\bar{\mathbf{x}}_0 + \alpha_{02}\mathbf{K}\bar{\mathbf{x}}_2 + \alpha_{02}\mathbf{K}\bar{\mathbf{x}}_0 = \alpha_{02}\mathbf{F}$$

Clearly, the Krylov series arise for the even terms of the deformation $\bar{\mathbf{x}}_{2n}$:

$$\bar{\mathbf{x}}_{2n} = (-1)^n \left(\mathbf{K}^{-1}\mathbf{M}\right)^n \bar{\mathbf{x}}_0$$

since the lowest-order term satisfies the static-deformation solution: $\mathbf{K}\bar{\mathbf{x}}_0 = \mathbf{F}$. The odd terms are zero: $\bar{\mathbf{x}}_{2n+1} = 0$.

6.1 Input dependence

In the previous section we have seen that the expansion of the force input in powers of the time t leads to correction terms, due to the dynamics of the deformation, of the form: $\mathbf{x}_n = (\mathbf{K}^{-1}\mathbf{M})^n\mathbf{x}_0$. With this knowledge, an expansion in terms of $\mathbf{x}_n$ is expected to converge fast for arbitrary input. Furthermore, the equation of motion can be projected onto these mode shapes with limited effort, resulting in an $A \times A$ dimensional equation of motion for the coefficients α_i:

$$\sum_{i,j=0}^{A} m_{ij}\ddot{\alpha}_j + d_{ij}\dot{\alpha}_j + k_{ij}\alpha_j = 0$$

with $\alpha_0(t)$ given, and $k_{ij} = m_{i(j-1)}$. Separately, there is an equation of motion for the rigid-body mode $\mathbf{q}$, in the operation direction:

$$\mathbf{q}^T\mathbf{M}\ddot{\mathbf{q}} + \mathbf{q}^T\mathbf{D}\dot{\mathbf{q}} = \mathbf{q}^T\mathbf{F}$$

Since $\mathbf{K}\mathbf{q} = 0$. The coupling occurs via the static-deformation equation, which defines the deformation $\mathbf{x}_0$:

$$\mathbf{K}\alpha(t)\mathbf{x}_0 = \mathbf{F} - \mathbf{M}\ddot{\mathbf{q}} - \mathbf{q}^T\mathbf{D}\dot{\mathbf{q}}$$

with the normalization $\mathbf{q}^T\mathbf{x}_0 = 0$, or, more generally, $\{\mathrm{Ker}K\} \perp \mathbf{x}_0$.

What has been considered a single input so far, could be multiple input as well:

$$\mathbf{F}(t) = \sum_j \alpha^j(t)\mathbf{F}_j$$

Each input generates a collection of A_j modes $\mathbf{x}_i^j$, to be included in the dynamics of α_i^j's. They can interact among each other through the the overlap $\mathbf{x}_i^{jT}\mathbf{M}\mathbf{x}_l^k$, $\mathbf{x}_i^{jT}\mathbf{D}\mathbf{x}_l^k$, and $\mathbf{x}_i^{jT}\mathbf{K}\mathbf{x}_l^k$. However, we should be careful not define some general theory, not tuned toward the operational mode. For example, take some lever transferring force between one end $\mathbf{F}_i$ and the other end $\mathbf{F}_o$. It would come natural to assume to ends as independent input $\mathbf{F}_i$ and $\mathbf{F}_o$. However, a lever is generally stiff and the rigid body mode would be a single equation of motion:

$$m\ddot{\mathbf{q}} = \mathbf{F}_o - \mathbf{F}_i = \mathbf{F}$$

where m is the sum mass of the lever. Hence, rather than two input, one should consider it a one-input problem for the internal dynamics. In the case of complex motion, it might be necessary to have multiple input, but then for different time intervals:

$$\mathbf{F}(t) = \begin{cases} \alpha(t)\mathbf{F}_1 & , \quad t_0 < t < t_1 \\ \alpha(t)\mathbf{F}_2 & , \quad t_1 < t < t_2 \\ \alpha(t)\mathbf{F}_3 & , \quad t_2 < t < t_3 \\ \quad etc. & \end{cases}$$

In such cases, it might be important to maintain some lower modes $\mathbf{x}_0^j(,\mathbf{x}_1^j,\cdots)$ in the next region $j+1$, since in the transition region around $t = t_j$ these modes might not immediately disappear in the absence of the static-deformation force $\mathbf{F}_j$.

7 Energy

So far we considered only the input force $\mathbf{F}$ not the corresponding displacement, except for the rigid-body motion. Part of the energy is converted to generate internal deformation, this corresponds to the elastic energy:

$$E_{\text{elastic}} = \frac{1}{2}\mathbf{x}_0^T\mathbf{K}\mathbf{x}_0$$

where $\mathbf{x}_0$ is the internal deformation in the static-deformation case. The definition of the static-deformation state, for constant acceleration, was such that the rigid-body motion under constant force performed a constant acceleration. In the case of the instantaneous approximation, the trajectory is the time-integral of static-deformation instances. A more rigid object would have less deformation, and it would therefore store less elastic energy when submitted to the same constant force. The kinetic, rigid-body energy, however, would be the same. The total energy is the sum of the elastic energy, which depends only on the deformation and the force $\mathbf{F}(t)$ at a given time, and the kinetic energy which depend on the actual trajectory, i.e., the history of the force input.

$$\begin{aligned} E_{\text{total}}(t) &= E_{\text{elastic}}(\mathbf{x}_0) + E_{\text{kinetic}}(t) \\ &= \frac{1}{2}\mathbf{x}_0^T \mathbf{K} \mathbf{x}_0 + \frac{1}{2}\dot{\mathbf{q}}^T(t)\mathbf{M}\dot{\mathbf{q}}(t) \end{aligned}$$

where the coefficient $\alpha(t)$ of the rigid-body motion $\mathbf{q}(t) = \alpha(t)\mathbf{q}$ satisfies the rigid-body differential equation:

$$\mathbf{M}\mathbf{q}\ddot{\alpha} + \mathbf{D}\mathbf{q}\dot{\alpha} = \mathbf{F}_{\parallel}$$

which a $3N$ copies of the same equation, which can be reduced to a single equation by projecting on $\mathbf{q}^T$. The parallel force depends on the ratio of the inertial and the resistive force, as we have seen in the instantaneous case. The total energy supplied to the system is larger, as energy is dissipated through the damping term $\mathbf{D}$. The total power P transfer as result of the input force is given by:

$$P(t) = \mathbf{F}^T(t)\left(\dot{\mathbf{q}}(t) + \dot{\mathbf{x}}_0\right)$$

where $\mathbf{F}_{\parallel}$ is part of the dynamical rigid-body equation, and the remainder $\mathbf{F}_{\perp}$ appears in the instantaneous deformation equation for $\mathbf{x}_0$:

$$P = \mathbf{F}_{\parallel}^T \dot{\mathbf{q}} + \mathbf{F}_{\perp}^T \dot{\mathbf{x}}_0$$

Inserting the equation of motion for $\mathbf{q}$ and the balance equation for $\mathbf{x}_0$ in the power equation:

$$\begin{aligned} P &= \dot{\mathbf{q}}^T \mathbf{M}\ddot{\mathbf{q}} + \mathbf{x}_0^T \mathbf{K}\dot{\mathbf{x}}_0 + \dot{\mathbf{x}}_0^T(\mathbf{M}\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}}) + \dot{\mathbf{q}}^T \mathbf{D}\dot{\mathbf{q}} \\ &= \dot{E}_{\text{total}} + \dot{\mathbf{x}}_0^T \mathbf{F}_{\text{intern}} + \dot{\mathbf{q}}^T \mathbf{D}\dot{\mathbf{q}} \end{aligned}$$

where the last term is the classical dissipation term. The internal force $\mathbf{F}_{\text{intern}}$ is the rigid-body response to the applied force. In a sense, it is the internal balancing force, which makes the not-so rigid body act coherently. In the absence of an external force, $\mathbf{F}_{\text{intern}} = 0$. Due to the internal deformation, the second-order system turns into a higher-order system, where the order $2n + 2$ is determined by the number of modes $\mathbf{x}_n$ taken into account.

8 Conclusions

In this paper, we started out with a number of rigid-body motions which occur in actual situations, such as a constant acceleration due to a constant force, and a constant velocity under friction. In all these cases a static deformation occurs. The static deformation has been treated as the starting point of the internal dynamics, such that only the vibrations occur which are actually driven by the rigid-body, or operational, motion of the system. Each higher-order term in the expansion arises from taking into account the inertial effects of the dynamics of the previous term, yielding a fast convergence. This is the basis for a model reduction, which allows one the study complex dynamics of FEM systems, with arbitrary input, with a limited number of degrees of freedom, without introducing model bias, beyond the choice of operation conditions, such as the position of the fixtures and desired operation trajectories.